

# INFRARED BOUND AND MEAN-FIELD BEHAVIOUR IN THE QUANTUM ISING MODEL

JAKOB E. BJÖRNBERG

ABSTRACT. We prove an infrared bound for the transverse field Ising model. This bound is stronger than the previously known infrared bound for the model, and allows us to investigate mean-field behaviour. As an application we show that the critical exponent  $\gamma$  for the susceptibility attains its mean-field value  $\gamma = 1$  in dimension at least 4 (positive temperature), respectively 3 (ground state), with logarithmic corrections in the boundary cases.

## 1. INTRODUCTION

Infrared bounds were originally developed in the 1970's as a method for proving the existence of phase transitions [12, 13, 14]. They were subsequently also used to establish mean-field behaviour in high-dimensional spin systems, in the sense that certain critical exponents attain their mean-field values [1, 2, 3, 20]. For establishing phase transitions the method is applicable to models where the spins have a continuous symmetry group, such as the classical Heisenberg and  $O(n)$  models; see [8] for a review. In proving mean-field behaviour, the infrared bound is useful because it implies the finiteness of the 'bubble diagram' in sufficiently high dimension. Similar bounds appear also in the analysis of high-dimensional percolation models [16].

The method of infrared bounds was first developed for classical spin systems, but it was quickly extended to quantum models. In the quantum setting, the method was successful for proving the existence of a phase transition for a range of models, including the Heisenberg antiferromagnet [12]. However, no attention has yet been paid to its implications for mean-field behaviour in quantum models. The main objective of this article is to establish an infrared bound which can be used to investigate mean-field behaviour in the quantum setting.

The model we will consider in this article is the *transverse field Ising model* on the integer lattice  $\mathbb{Z}^d$ . Let  $\Lambda \subseteq \mathbb{Z}^d$  be finite. In the language of quantum spin systems, the transverse field Ising model in the volume

---

*Date:* January 9, 2013.

Department of Mathematics, Uppsala University, Box 256, 751 05 Uppsala, Sweden, Phone +46(0)18-471 3106, e-mail: jakob@math.uu.se.

$\Lambda$  has Hamiltonian

$$(1) \quad H_\Lambda = -\lambda \sum_{xy \in \Lambda} \sigma_x^{(3)} \sigma_y^{(3)} - \delta \sum_{x \in \Lambda} \sigma_x^{(1)}$$

acting on the Hilbert space  $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathbb{C}^2$ . Here the first sum is over all (unordered) nearest neighbour sites in  $\Lambda$  and the second sum is over all single sites;

$$\sigma^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \sigma^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

are the spin- $\frac{1}{2}$  Pauli matrices; and  $\lambda$  and  $\delta$  denote the spin-coupling and transverse field intensities, respectively. The model was introduced in [18] and has been widely studied since.

Let  $\beta > 0$  be a fixed real number (known as the ‘inverse temperature’) and define the finite volume, positive temperature state  $\langle \cdot \rangle_{\Lambda, \beta}$  by

$$(2) \quad \langle Q \rangle_{\Lambda, \beta} = \frac{\text{tr}(e^{-\beta H_\Lambda} Q)}{\text{tr}(e^{-\beta H_\Lambda})},$$

where  $Q$  is a suitable observable (matrix). One may then define the finite-volume *ground state*  $\langle \cdot \rangle_{\Lambda, \infty}$ , as well as *infinite-volume* states  $\langle \cdot \rangle_\beta$  by

$$\langle \cdot \rangle_{\Lambda, \infty} = \lim_{\beta \uparrow \infty} \langle \cdot \rangle_{\Lambda, \beta}, \quad \langle \cdot \rangle_\beta = \lim_{\Lambda \uparrow \mathbb{Z}^d} \langle \cdot \rangle_{\Lambda, \beta}, \quad \text{for } 0 < \beta \leq \infty.$$

See for example [5] for more information about this. It is known [15] that for each  $\delta > 0$  and  $0 < \beta \leq \infty$  there is a critical value  $\lambda_c = \lambda_c(\delta, \beta)$  of  $\lambda$  which marks the point below which the *two-point correlation function*

$$(3) \quad c(x, y) = \langle \sigma_x^{(3)} \sigma_y^{(3)} \rangle_\beta$$

vanishes as  $|x - y| \rightarrow \infty$ , and above which it does not. This critical point may also be characterized in terms of the vanishing/non-vanishing of the spontaneous  $\sigma^{(3)}$ -magnetization, or of the finiteness/non-finiteness of the susceptibility [10].

For the classical Ising model ( $\delta = 0$  and  $\lambda = 1$  in (1)), the infrared bound is an upper bound on the Fourier transform

$$\hat{c}(k) = \sum_{x \in \mathbb{Z}^d} c(0, x) e^{ik \cdot x}, \quad k \in (-\pi, \pi]^d$$

of the two-point function (3). Here  $k \cdot x$  denotes the usual scalar product in  $\mathbb{R}^d$ . The classical infrared bound states that

$$(4) \quad \hat{c}(k) \leq \frac{1}{2\beta \hat{L}(k)},$$

where  $\hat{L}(k) = \sum_{j=1}^d (1 - \cos(k_j))$  is the Fourier transform of the Laplacian on  $\mathbb{Z}^d$ . For quantum models, infrared bounds are typically stated

in terms of the *Duhamel two-point function*

$$(5) \quad b(x) = (\sigma_0^{(3)}, \sigma_x^{(3)}) := \frac{1}{\text{tr}(e^{-\beta H_\Lambda})} \int_0^\beta \text{tr}(e^{-(\beta-t)H_\Lambda} \sigma_x^{(3)} e^{-tH_\Lambda} \sigma_0^{(3)}) dt$$

rather than the usual two-point function (3). For the transverse field Ising model, it follows as a special case of [12, Theorem 4.1] that

$$(6) \quad \hat{b}(k) \leq \frac{1}{2\lambda \hat{L}(k)}.$$

(Note that our  $\hat{b}(k)$  and  $H_\Lambda$  differ from the quantities in [12] by factors  $\beta$  and  $\lambda$ , respectively.)

The main result of this article, Theorem 1.2, strengthens (6). This will allow us to compute the critical exponent  $\gamma$  for the susceptibility in dimension  $d \geq 4$  (finite temperature) respectively  $d \geq 3$  (ground state); see Theorem 1.3. Before stating the main result we describe a graphical representation of the transverse field Ising model which is of fundamental importance to our analysis.

**1.1. Graphical representation.** It is well-known [5, 6] that the transverse field Ising model on  $\Lambda$  possesses a ‘path integral representation’ which expresses it as a type of classical Ising model on the continuous space  $\Lambda \times [0, \beta)$ . This may be expressed as follows. In what follows  $\beta < \infty$ , although conclusions about the ground state may be obtained by letting  $\beta \rightarrow \infty$ .

Let  $[0, \beta)^{\mathfrak{p}}$  denote the circle of length  $\beta$ , formally defined as  $[0, \beta)^{\mathfrak{p}} = \{e^{2\pi it/\beta} : t \in \mathbb{R}\}$ . Usually we identify  $[0, \beta)^{\mathfrak{p}}$  with its parameterization for  $t \in [0, \beta)$ ; the superscript  $\mathfrak{p}$  serves as a reminder that the set is ‘periodic’. We let  $N$  be an integer and throughout the article let  $\Lambda = \Lambda_N = (\mathbb{Z}/2N)^d$  be the  $d$ -dimensional torus of side  $2N$ .

Let  $E[\cdot]$  denote a probability measure governing the following:

- (1) a collection  $D = (D_x : x \in \Lambda)$  of independent Poisson processes of intensity  $\delta$  on  $[0, \beta)$ , conditioned on the number of points  $|D_x|$  being even for each  $x \in \Lambda$ ; and
- (2) a collection  $\xi = (\xi_x : x \in \Lambda)$  of independent random variables, taking values 0 or 1 with probability 1/2 each,

such that  $D$  and  $\xi$  are independent of each other. Write  $\sigma(x, t) = (-1)^{\xi_x + |D_x \cap [0, t]|}$  for the right-continuous function of  $t$  which changes between  $-1$  and  $+1$  at the points of  $D_x$  and takes the value  $(-1)^{\xi_x}$  at  $t = 0$ . Note that  $\sigma$  is well-defined as a function on  $[0, \beta)^{\mathfrak{p}}$ . Write  $\Sigma_\Lambda$  for the set of such  $\sigma(x, t)$ . This set may be endowed with a natural sigma-field  $\mathcal{F}_\Lambda$ , generated by the projections

$$(7) \quad \sigma \mapsto (\sigma(x_1, t_1), \dots, \sigma(x_n, t_n)) \quad \text{for } n \geq 1, x_j \in \Lambda, t_j \in [0, \beta).$$

Let

$$(8) \quad Z_\Lambda^\beta = E \left[ \exp \left( \lambda \sum_{x \sim y} \int_0^\beta \sigma(x, t) \sigma(y, t) dt \right) \right],$$

where the sum is over nearest neighbours  $x, y \in \Lambda$ .

DEFINITION 1.1. *The space–time Ising measure  $\mu_\Lambda^\beta$  is the probability measure on  $(\Sigma_\Lambda, \mathcal{F}_\Lambda)$  given by*

$$\mu_\Lambda^\beta(f) = \frac{1}{Z_\Lambda^\beta} E \left[ f(\sigma) \exp \left( \lambda \sum_{x \sim y} \int_0^\beta \sigma(x, t) \sigma(y, t) dt \right) \right]$$

for each bounded, measurable test function  $f : \Sigma_\Lambda \rightarrow \mathbb{R}$ .

(Here, and in what follows,  $\mu_\Lambda^\beta(f)$  denotes the expectation of  $f$  under the measure  $\mu_\Lambda^\beta$ .) The measure  $\mu_\Lambda^\beta$  corresponds with the state  $\langle \cdot \rangle_{\Lambda, \beta}$  in the sense that for any finite set  $A \subseteq \Lambda$ , we have the identity

$$(9) \quad \left\langle \prod_{x \in A} \sigma_x^{(3)} \right\rangle_{\Lambda, \beta} = \mu_\Lambda^\beta \left( \prod_{x \in A} \sigma(x, 0) \right).$$

In very brief terms, this correspondence is based on applying the Lie–Trotter product formula to the operator  $e^{-\beta H_\Lambda}$  and evaluating the trace in the  $\sigma^{(3)}$ -basis. See [5] and references therein, and also [17] for information about similar graphical representations. There exist limits of the measures  $\mu_\Lambda^\beta$  as  $N \rightarrow \infty$  and/or  $\beta \rightarrow \infty$ , and a suitable version of (9) holds in infinite volume, also for the ground state.

**1.2. Main results.** We now outline the main results of this article, saving more detailed statements and explanations for the relevant later sections. Our main results concern the *Schwinger function*:

$$(10) \quad c_\Lambda^\beta((x, s), (y, t)) := \mu_\Lambda^\beta(\sigma(x, s) \sigma(y, t)), \quad (0 \leq s, t < \beta).$$

Note that this may alternatively be expressed as

$$c_\Lambda^\beta((x, s), (y, t)) = \frac{1}{\text{tr}(e^{-\beta H_\Lambda})} \text{tr}(e^{-(\beta-t+s)H_\Lambda} \sigma_y^{(3)} e^{-(t-s)H_\Lambda} \sigma_x^{(3)}),$$

as may be seen using a Lie–Trotter expansion as for (9); cf. (2) and (5). For  $k \in \frac{2\pi}{2N}\Lambda$  and  $l \in \frac{2\pi}{\beta}\mathbb{Z}$  let

$$\hat{c}_\Lambda^\beta(k, l) = \sum_{x \in \Lambda} \int_0^\beta c_\Lambda^\beta((0, 0), (x, t)) e^{ik \cdot x} e^{ilt} dt$$

be the Fourier transform of (10). The following is the main result of this article; it holds for finite volume and positive temperature, but has implications for infinite volume and ground state which we discuss below.

THEOREM 1.2 (Infrared bound). *Let  $k \in \frac{2\pi}{2N}\Lambda$  and  $l \in \frac{2\pi}{\beta}\mathbb{Z}$ . Then*

$$\hat{c}_\Lambda^\beta(k, l) \leq \frac{48}{2\lambda\hat{L}(k) + l^2/2\delta}.$$

This result is proved in Section 2. In fact, we will prove the slightly stronger bound

$$\hat{c}_\Lambda^\beta(k, l) \leq \frac{2\lambda\hat{L}(k) + 48l^2/2\delta}{(2\lambda\hat{L}(k) + l^2/2\delta)^2},$$

see (26). Setting  $l = 0$  gives

$$(11) \quad \sum_{x \in \Lambda} \left( \int_0^\beta \mu_\Lambda^\beta(\sigma(0, 0)\sigma(x, t)) dt \right) e^{ik \cdot x} \leq \frac{1}{2\lambda\hat{L}(k)}.$$

The integral in (11) equals the Duhamel two-point function  $b(x) = (\sigma_0^{(3)}, \sigma_x^{(3)})$  of (5). Thus (11) is [12, Theorem 4.1] in the special case of the transverse field Ising model. It also reduces to the classical bound (4) when  $\delta = 0$  and  $\lambda = 1$ . We now describe some applications of Theorem 1.2.

Let  $0 < \beta \leq \infty$  be fixed. For each  $\lambda < \lambda_c$  there is a unique infinite-volume limit  $\mu^\beta$  of the measures  $\mu_\Lambda^\beta$ . We let

$$(12) \quad \chi = \chi(\delta, \lambda, \beta) := \sum_{x \in \mathbb{Z}^d} \int_{-\beta/2}^{\beta/2} \mu^\beta(\sigma(0, 0)\sigma(x, t)) dt$$

denote the *susceptibility*. (We have chosen the bounds  $-\beta/2, \beta/2$  in the integral rather than  $0, \beta$  to get the correct range also for the case  $\beta = \infty$ ; for  $\beta < \infty$  the two choices are equivalent.) To motivate this choice of name, note that

$$\begin{aligned} \sum_{x \in \Lambda} \int_{-\beta/2}^{\beta/2} \mu_\Lambda^\beta(\sigma(0, 0)\sigma(x, t)) dt &= \sum_{x \in \Lambda} (\sigma_0^{(3)}, \sigma_x^{(3)}) \\ &= \frac{d}{d\nu} \left[ \frac{\text{tr}(e^{-\beta H_\Lambda^\nu} \sigma_0^{(3)})}{\text{tr}(e^{-\beta H_\Lambda^\nu})} \right]_{\nu=0} \end{aligned}$$

where

$$H_\Lambda^\nu = -\lambda \sum_{xy \in \Lambda} \sigma_x^{(3)} \sigma_y^{(3)} - \delta \sum_{x \in \Lambda} \sigma_x^{(1)} - \nu \sum_{x \in \Lambda} \sigma_x^{(3)}.$$

It is known that  $\chi$  is finite for  $\lambda < \lambda_c$  and diverges as  $\lambda \uparrow \lambda_c$ ; see [10]. The critical exponent  $\gamma$  may be defined by the expected critical behaviour

$$\chi(\lambda) \sim (\lambda_c - \lambda)^{-\gamma} \quad \text{as } \lambda \uparrow \lambda_c \text{ with } \delta \text{ fixed.}$$

For the classical Ising model, it was proved in [1, 3] that  $\gamma$  exists and equals 1 for  $d \geq 4$  (with logarithmic corrections when  $d = 4$ ). One may also consider the speed of divergence of  $\chi(\delta, \lambda)$  as  $(\delta, \lambda)$  approaches the critical curve  $(\delta, \lambda_c(\delta))$  along any straight line. As an application

of Theorem 1.2, we will prove the following result. We let  $\rho(\delta, \lambda) = \sqrt{(\delta - \delta_0)^2 + (\lambda - \lambda_c(\delta_0))^2}$  denote the distance from  $(\delta, \lambda)$  to a specified point  $(\delta_0, \lambda_c(\delta_0))$  on the critical curve.

**THEOREM 1.3.** *Let  $\delta_0 > 0$  and let  $(\delta, \lambda)$  approach  $(\delta_0, \lambda_c(\delta_0))$  along any straight line strictly inside the quadrant  $\{(\delta, \lambda) : \delta > \delta_0, \lambda < \lambda_c(\delta_0)\}$ . There are finite constants  $c_1, c_2$ , depending on  $d, \beta$  and the slope of the line of approach, such that the following holds.*

- (1) *Suppose that either  $\beta < \infty$  and  $d > 4$ , or  $\beta = \infty$  and  $d > 3$ . Then*

$$c_1 \rho(\delta, \lambda)^{-1} \leq \chi(\delta, \lambda) \leq c_2 \rho(\delta, \lambda)^{-1}$$

*as  $\rho(\delta, \lambda) \downarrow 0$ .*

- (2) *Suppose that either  $\beta < \infty$  and  $d = 4$ , or  $\beta = \infty$  and  $d = 3$ . Then*

$$c_1 \rho(\delta, \lambda)^{-1} \leq \chi(\delta, \lambda) \leq -c_2 \log \rho(\delta, \lambda) / \rho(\delta, \lambda)$$

*as  $\rho(\delta, \lambda) \downarrow 0$ .*

Theorem 1.3 is proved in Section 3; in fact we prove slightly more, see Propositions 3.3 and 3.5. In proving Theorem 1.3 we are led to study the *bubble-diagram*

$$(13) \quad B = B(\delta, \lambda, \beta) = \sum_{x \in \mathbb{Z}^d} \int_{-\beta/2}^{\beta/2} \mu^\beta(\sigma(0, 0) \sigma(x, t))^2 dt.$$

(The term ‘bubble-diagram’ comes from analogy with a related quantity which appears in the study of mean-field behaviour in the classical Ising model [1, 2, 3].) Theorem 1.2, together with the Plancherel identity, allows us to deduce upper bounds on  $B$  (Lemma 3.2). In addition to such bounds we also require new differential inequalities (Lemma 3.1). The method of proof would also give the critical exponent  $\gamma = 1$  for approach to criticality along lines with constant  $\delta$  or constant  $\lambda$ , *subject to* first proving additional differential inequalities. See Remark 3.6.

**1.3. Discussion.** It is interesting to note that we obtain the ‘classical’ critical exponent value  $\gamma = 1$  also at the ‘quantum critical point’  $\beta = \infty, \lambda = \lambda_c(\delta)$ . Our analysis only deals with approach to criticality with temperature (hence  $\beta$ ) kept fixed, so our results do not rule out the possibility of a different critical exponent value for approach to the quantum critical point with varying temperature, as described in [19].

In the classical Ising model, the type of methods used in this article can only give conclusions about critical exponents down to dimension  $d = 4$ . We are able to obtain conclusions about the case  $d = 3$  essentially because the ‘imaginary time representation’ described in Section 1.1 maps the quantum Ising model in  $d$  dimensions onto a classical

model in  $d + 1$  dimensions. This intuition is in some sense only valid in the case  $\beta = \infty$  when the ‘imaginary time axis’ is unbounded.

In the classical Ising model it was also possible to use the infrared bound and differential inequalities to determine critical exponents for the magnetization [2]. It is to be expected that Theorem 1.2 can be used to obtain similar results also for the quantum Ising model.

Note that the arguments used to prove Theorem 1.3 require bounds on  $B$  as defined above, *not* on the (one might think more natural) quantity

$$\sum_{x \in \mathbb{Z}^d} \langle \sigma_0^{(3)} \sigma_x^{(3)} \rangle_\beta^2 = \sum_{x \in \mathbb{Z}^d} \mu^\beta (\sigma(0, 0) \sigma(x, 0))^2,$$

which does not feature in this work.

It is worth remarking that we approach Theorem 1.2 by working directly in the continuous set-up of Definition 1.1. A natural alternative would be to work with the discrete approximation of the partition function (8) inherent in the Lie–Trotter product formula on which Definition 1.1 is based. In this way certain technicalities associated with working with continuous ‘time’ may be avoided; on the other hand other issues to do with discrete approximation would be introduced. This discrete approximation is closely related to a way of expressing the space–time Ising model as a (weak) limit of classical Ising models (cf. [9, Section 2.2.2]), each of which obeys a bound of the form (4). Unfortunately, simply taking the limit in the corresponding bound gives only (6) and not the full bound of Theorem 1.2.

Finally, although we have chosen to focus entirely on the nearest-neighbour transverse field Ising model, it seems likely that our arguments can be extended to other reflection positive models. For example, it seems straightforward to extend Theorem 1.2 to other interactions than nearest neighbour, such as the Yukawa and power law potentials described in [8, Section 3]. Many of the arguments in Section 2 apply when the definition  $\sigma(x, t) = (-1)^{\xi_x + |D_x \cap [0, t]|}$  in the graphical representation (8) is modified to  $\sigma(x, t) = \psi_x (-1)^{|D_x \cap [0, t]|}$  for more general  $\psi_x \in \mathbb{R}^n$ . The proofs of the differential inequalities in Section 3, and hence Theorem 1.3, rely on the ‘random-parity representation’ (a relative of the random-current representation) and so are quite Ising-specific. However, certain extensions of these results to more general translation-invariant interactions are most likely possible, as for the classical case treated in [1].

**Acknowledgements.** The author would like to thank the following. Geoffrey Grimmett for introducing him to the subject, and for many valuable discussions in the early phases of the project. Svante Janson for valuable feedback on a draft of this article, and in particular for suggesting an improved proof of Lemma 2.6. And finally the anonymous referees for many insightful and helpful comments.

## 2. THE INFRARED BOUND

In this section we prove the main result of this article, Theorem 1.2. First we present more detailed notation.

We write  $\mathbb{I}\{A\}$  for the indicator of the event  $A$ , taking value 1 if  $A$  occurs and 0 otherwise. Let  $N \geq 1$  be an integer, and let  $\Lambda = (\mathbb{Z}/2N)^d$  be a torus in  $d$  dimensions. Thus  $\Lambda$  is the graph whose vertex set is the set of vectors  $x = (x_1, \dots, x_d) \in \{0, 1, \dots, 2N-1\}^d$ , and whose adjacency relation  $\sim$  is given by:  $x \sim y$  if there is  $j \in \{1, \dots, d\}$  such that (a)  $x_j - y_j \equiv 1 \pmod{2N}$ , and (b)  $x_k = y_k$  for all  $k \neq j$ . Write  $L$  for the *graph Laplacian* of  $\Lambda$ :

$$(14) \quad L(x, y) = d\mathbb{I}\{x = y\} - \frac{1}{2}\mathbb{I}\{x \sim y\}, \quad x, y \in \Lambda.$$

We write  $Lu$  for the function  $\Lambda \rightarrow \mathbb{C}$  given by the matrix-vector product

$$(15) \quad (Lu)(x) = \sum_{y \in \Lambda} L(x, y)u(y).$$

For  $u, v : \Lambda \rightarrow \mathbb{C}$  we write  $\langle u, v \rangle$  for the usual vector inner product,

$$(16) \quad \langle u, v \rangle = \sum_{x \in \Lambda} u(x)v(x) \in \mathbb{C}.$$

Note that we do not conjugate the second argument.

Throughout this section  $\beta$  will be finite and fixed. Recall that  $[0, \beta)^{\mathfrak{p}}$  denotes the circle of length  $\beta$ . A function  $f : [0, \beta)^{\mathfrak{p}} \rightarrow \mathbb{R}$  can be thought of either as a function  $\mathbb{R} \rightarrow \mathbb{R}$  which is periodic with period  $\beta$ , or simply as a function  $[0, \beta) \rightarrow \mathbb{R}$ . We will usually take the latter viewpoint, taking care to remember, for example, that  $f$  is continuous only if, in particular, the limits  $\lim_{t \uparrow \beta} f(t)$  and  $\lim_{t \downarrow 0} f(t)$  exist and are equal, and similarly for differentiability and other analytic properties. The antiderivative of a measurable  $f : [0, \beta)^{\mathfrak{p}} \rightarrow \mathbb{R}$  is the function  $F : [0, \beta)^{\mathfrak{p}} \rightarrow \mathbb{R}$  given by

$$F(t) = \int_0^t f(s) ds \text{ for } t \in [0, \beta).$$

If  $\mathbf{h}$  is a function  $\Lambda \times [0, \beta)^{\mathfrak{p}} \rightarrow \mathbb{C}$ , we will often use the notation  $\mathbf{h} = (h(x, t) : x \in \Lambda, t \in [0, \beta)^{\mathfrak{p}})$ . For each  $x \in \Lambda$  we then write  $h(x, \cdot)$  for the function  $t \mapsto h(x, t)$ , and for each  $t \in [0, \beta)^{\mathfrak{p}}$  we write  $h(\cdot, t)$  for the function  $x \mapsto h(x, t)$ . Note that  $h(\cdot, t) : \Lambda \rightarrow \mathbb{C}$ , so the notation in (15) and (16) applies to  $h(\cdot, t)$ . We say that a function  $\mathbf{h} : \Lambda \times [0, \beta)^{\mathfrak{p}} \rightarrow \mathbb{R}$  is *bounded* if each  $h(x, \cdot)$  is bounded, *differentiable* if each  $h(x, \cdot)$  is differentiable, and similarly for other analytic properties. We write  $h'(x, t) = \frac{d}{dt}h(x, t)$  etc.

The measure  $E[\cdot]$ , which was used to define the space-time Ising measure  $\mu_{\Lambda}^{\beta}$  in Section 1.1, may be written as a product  $E = E_{\times} \times E_0$ . Here  $E_{\times}[\cdot]$  denotes a probability measure governing the collection  $D = (D_x : x \in \Lambda)$  of Poisson processes conditioned to have even size, and



$E_0[\cdot]$  denotes a probability measure governing the random variables  $\xi_x \in \{0, 1\}$  (for  $x \in \Lambda$ ). Recall that  $\sigma(x, t) = (-1)^{\xi_x + |D_x \cap [0, t]|}$  and that

$$Z_\Lambda^\beta = E \left[ \exp \left( \lambda \sum_{x \sim y} \int_0^\beta \sigma(x, t) \sigma(y, t) dt \right) \right].$$

Note that if  $u : \Lambda \rightarrow \mathbb{R}$ , then

$$(17) \quad \langle Lu, u \rangle = \frac{1}{2} \sum_{x \sim y} (u(x) - u(y))^2.$$

Since  $\sigma(x, t)^2 = 1$  for all  $x \in \Lambda$ ,  $t \in [0, \beta]^\mathfrak{p}$ , it follows that  $Z_\Lambda^\beta$  is a multiple of

$$(18) \quad Z(0) := E \left[ \exp \left( - \lambda \int_0^\beta \langle L\sigma(\cdot, t), \sigma(\cdot, t) \rangle dt \right) \right].$$

In fact,

$$\mu_\Lambda^\beta(f) = \frac{1}{Z(0)} E \left[ f(\sigma) \exp \left( - \lambda \int_0^\beta \langle L\sigma(\cdot, t), \sigma(\cdot, t) \rangle dt \right) \right].$$

The quantity  $Z(0)$  is a special case of  $Z(\mathbf{h})$  as defined in (19) below. Since  $\beta$  and  $\Lambda$  are fixed in what follows, we will suppress them from the notation  $\mu_\Lambda^\beta$  and simply write  $\mu$ . However, we will still write  $Z_\Lambda^\beta$  to distinguish it from  $Z(0)$ .

For a function  $f : \Lambda \times [0, \beta]^\mathfrak{p} \rightarrow \mathbb{R}$ , recall that the Fourier transform  $\hat{f}$  is given by

$$\hat{f}(k, l) = \sum_{x \in \Lambda} \int_0^\beta f(x, t) e^{ik \cdot x} e^{ilt} dt, \quad k \in \frac{2\pi}{2N} \Lambda, l \in \frac{2\pi}{\beta} \mathbb{Z}.$$

Throughout this section we will write  $c(x, t)$  for the Schwinger two-point function,

$$c(x, t) := \mu(\sigma(0, 0) \sigma(x, t)), \quad x \in \Lambda, t \in [0, \beta]^\mathfrak{p},$$

given in (3). Note that  $\hat{c}(k, l) \geq 0$  (indeed,  $\hat{c}(k, l) = \mu[|\hat{\sigma}(k, l)|^2]/(\beta|\Lambda|)$ ).

The process  $D$  is the set of discontinuities of  $\sigma$ . Under  $\mu$  it is absolutely continuous with respect to a Poisson process of intensity  $\delta$  on  $\Lambda \times [0, \beta]^\mathfrak{p}$ , but the density depends on  $\Lambda$ . The following lemma gives a ‘uniform stochastic bound’ on  $D$ . For two point processes  $C$  and  $D$  on  $\Lambda \times [0, \beta]^\mathfrak{p}$  we say that  $D$  is *stochastically dominated* by  $C$  if there is a coupling  $\mathbb{P}$  of  $C$  and  $D$  such that  $\mathbb{P}(D \subseteq C) = 1$ .

**LEMMA 2.1.** *Under  $\mu$ , the process  $D$  is stochastically dominated by a Poisson process of intensity  $2\delta$ .*

*Proof.* The proof uses the space–time random-cluster (or FK-) representation, which is described in [9, Chapter 2]. (The space–time Ising measure is defined slightly differently in [9] than in the present work, but the equivalence of the definitions follows from elementary properties

of Poisson processes.) Let  $\phi_{q;\lambda,\delta}$  denote the space–time random-cluster measure on  $\Lambda \times [0, \beta]^{\mathfrak{p}}$ . As described in [9, Section 2.1], a realization of  $\sigma$  with law  $\mu$  can be obtained from a realization  $\omega$  with law  $\phi_{2;\lambda,\delta}$  by assigning to each connected component spin  $\pm 1$  independently with probability  $1/2$  each. Let  $C$  denote the process of ‘cuts’ in  $\omega$ . It follows that  $D \subseteq C$ . Moreover, by [9, Corollary 2.2.13], the process of cuts under  $\phi_{2;\lambda,\delta}$  is stochastically dominated by the process of cuts under  $\phi_{1;\lambda/2,2\delta}$ . Under the latter measure, the process of cuts is a Poisson process with intensity  $2\delta$ .  $\square$

We will prove Theorem 1.2 by establishing a variational result, which we describe in the next subsection.

**2.1. Gaussian domination.** For  $\mathbf{h}$  bounded and *twice* differentiable we define the quantity

$$(19) \quad Z(\mathbf{h}) = E \left[ \exp \left( -\lambda \int_0^\beta \langle L[\sigma(\cdot, t) + h(\cdot, t)], [\sigma(\cdot, t) + h(\cdot, t)] \rangle dt + \frac{1}{2\delta} \sum_{x \in \Lambda} \int_0^\beta h''(x, t) \sigma(x, t) dt \right) \right].$$

We will deduce Theorem 1.2 from an upper bound on  $Z(\mathbf{h})$ . The type of bound we will derive is similar to what is known as ‘Gaussian domination’, although we will not pursue any connections to Gaussian gradient models here.

If  $\mathbf{h}$  has the special property that there is a function  $h : [0, \beta]^{\mathfrak{p}} \rightarrow \mathbb{R}$  such that  $h(x, t) = h(t)$  for all  $x \in \Lambda$ , then we will write  $Z(h)$  for  $Z(\mathbf{h})$ . Using (17) we see that

$$Z(h) = E \left[ \exp \left( -\lambda \int_0^\beta \langle L\sigma(\cdot, t), \sigma(\cdot, t) \rangle dt + \frac{1}{2\delta} \sum_{x \in \Lambda} \int_0^\beta h''(t) \sigma(x, t) dt \right) \right].$$

Clearly the same expression is valid if each  $h(x, \cdot) = h(\cdot)$  almost everywhere. If, moreover,  $h''(t) = 0$  for all  $t \in [0, \beta]^{\mathfrak{p}}$  then it follows that  $Z(h) = Z(0)$  as given in (18).

The proof of Theorem 1.2 rests on two main lemmas, of which the following is the first:

**LEMMA 2.2.** *Let  $\mathbf{h}$  be twice differentiable. Then there is some  $z \in \Lambda$  such that, writing  $h(t)$  for  $h(z, t)$ , we have  $Z(\mathbf{h}) \leq Z(h)$ .*

Lemma 2.2 will be proved in Section 2.2.

Next, write  $|D| := \sum_{x \in \Lambda} |D_x|$  for the total number of points in  $D$ , and define

$$(20) \quad \zeta(r) = \mu[\cosh(r/\delta)^{|D|}], \quad r \in \mathbb{R}.$$

(Recall that we write  $\mu[\cdot]$  for expectation wrt  $\mu$ .) It follows from Lemma 2.1 that  $\zeta(r)$  is analytic in  $r \in \mathbb{R}$ , and hence

$$(21) \quad \zeta(r) = 1 + \frac{r^2}{2\delta^2} \mu|D| + O(r^4), \quad \text{as } r \rightarrow 0.$$

Note that

$$(22) \quad \mu|D| = |\Lambda| \mu|D_0| \leq 2\beta\delta|\Lambda|,$$

by translation invariance and Lemma 2.1.

For  $f : [0, \beta)^{\mathfrak{p}} \rightarrow \mathbb{R}$ , let

$$\|f\|_2 = \left( \int_0^\beta f(t)^2 dt \right)^{1/2} \quad \text{and} \quad \|f\|_\infty = \text{esssup}_{t \in [0, \beta)^{\mathfrak{p}}} |f(t)|$$

denote the  $L^2$ - and  $L^\infty$ -norms of  $f$ , respectively. The second main step in proving Theorem 1.2 is to establish the following result:

LEMMA 2.3. *Let  $h : [0, \beta)^{\mathfrak{p}} \rightarrow \mathbb{R}$  be twice differentiable. Then*

$$Z(h) \leq \zeta(\|h'\|_\infty) Z(0).$$

Lemma 2.3 will be proved in Section 2.3. Whereas Lemma 2.2 can be proved using arguments similar to those for previously known infrared bounds, Lemma 2.3 requires new ideas. We now show how Theorem 1.2 follows from Lemmas 2.2 and 2.3.

*Proof of Theorem 1.2.* Let  $\mathbf{h}$  be twice differentiable and let  $h(\cdot) = h(z, \cdot)$  be as in Lemma 2.2. We will prove the result by making a particular choice of  $\mathbf{h}$ , but for the time being we assume only that there is  $q \in (0, 1]$  such that the set  $\{t \in [0, \beta)^{\mathfrak{p}} : |h'(t)| \geq \|h'\|_\infty/2\}$  has Lebesgue measure at least  $q\beta$ . Then

$$\|h'\|_2^2 = \int_0^\beta h'(t)^2 dt \geq q\beta(\|h'\|_\infty/2)^2,$$

so by monotonicity of  $\zeta$  we have that

$$\zeta(\|h'\|_\infty) \leq \zeta\left(\frac{2}{\sqrt{q\beta}}\|h'\|_2\right).$$

Hence by Lemmas 2.2 and 2.3,

$$Z(\mathbf{h}) \leq Z(h) \leq \zeta\left(\frac{2}{\sqrt{q\beta}}\|h'\|_2\right) Z(0).$$

Using the fact that  $\langle L(h + \sigma), (h + \sigma) \rangle = 2\langle Lh, \sigma \rangle + \langle Lh, h \rangle + \langle L\sigma, \sigma \rangle$  and dividing by  $Z(0)$  it follows that

$$(23) \quad \begin{aligned} & \mu \left[ \exp \left( -2\lambda \int_0^\beta \langle Lh(\cdot, t), \sigma(\cdot, t) \rangle dt + \frac{1}{2\delta} \sum_{x \in \Lambda} \int_0^\beta h''(x, t) \sigma(x, t) \right) \right] \\ & \leq \zeta \left( \frac{2}{\sqrt{q}\beta} \|h'\|_2 \right) \exp \left( \lambda \int_0^\beta \langle Lh(\cdot, t), h(\cdot, t) \rangle dt \right). \end{aligned}$$

Replace  $\mathbf{h}$  by  $\alpha \mathbf{h} = (\alpha h(x, t) : x \in \Lambda, t \in [0, \beta]^p)$  for  $\alpha > 0$  and expand  $\zeta$  and the exponentials in (23) as power series. In the left-hand-side, the term of order  $\alpha$  as  $\alpha \rightarrow 0$  is

$$\mu \left[ -\lambda \int_0^\beta \langle Lh(\cdot, t), \sigma(\cdot, t) \rangle dt + \frac{1}{2\delta} \sum_{x \in \Lambda} \int_0^\beta h''(x, t) \sigma(x, t) \right] = 0$$

by the  $\pm$ -symmetry of  $\sigma$  under  $\mu$ . Using (21) we find, on comparing terms of order  $\alpha^2$ , that

$$(24) \quad \begin{aligned} & \frac{1}{2} \mu \left[ \left( 2\lambda \int_0^\beta \langle Lh(\cdot, t), \sigma(\cdot, t) \rangle dt - \frac{1}{2\delta} \sum_{x \in \Lambda} \int_0^\beta h''(x, t) \sigma(x, t) \right)^2 \right] \\ & \leq \frac{2\mu|D|}{\delta^2 q \beta} \int_0^\beta h'(t)^2 dt + \lambda \int_0^\beta \langle Lh(\cdot, t), h(\cdot, t) \rangle dt. \end{aligned}$$

Now let  $g(x, t) = a(x, t) + ib(x, t) \in \mathbb{C}$ , where  $a(\cdot, \cdot), b(\cdot, \cdot) : \Lambda \times [0, \beta]^p \rightarrow \mathbb{R}$  are twice differentiable. Assume each  $a(x, \cdot)$  and  $b(x, \cdot)$  satisfies the assumptions on  $h$  above, with the same  $q$ . Since

$$\begin{aligned} \langle Lg(\cdot, t), \sigma(\cdot, t) \rangle &= \langle La(\cdot, t), \sigma(\cdot, t) \rangle + i \langle Lb(\cdot, t), \sigma(\cdot, t) \rangle, \\ \langle Lg(\cdot, t), \overline{g(\cdot, t)} \rangle &= \langle La(\cdot, t), a(\cdot, t) \rangle + \langle Lb(\cdot, t), b(\cdot, t) \rangle, \\ g''(x, t) &= a''(x, t) + ib''(x, t), \text{ and} \\ |g'(x, t)|^2 &= a'(x, t)^2 + b'(x, t)^2, \end{aligned}$$

it follows from (24) that

$$(25) \quad \begin{aligned} & \mu \left[ \left| 2\lambda \int_0^\beta \langle Lg(\cdot, t), \sigma(\cdot, t) \rangle dt - \frac{1}{2\delta} \sum_{x \in \Lambda} \int_0^\beta g''(x, t) \sigma(x, t) \right|^2 \right] \\ & \leq \frac{4\mu|D|}{\delta^2 q \beta} \int_0^\beta (a'(z_1, t)^2 + b'(z_2, t)^2) dt + 2\lambda \int_0^\beta \langle Lg(\cdot, t), \overline{g(\cdot, t)} \rangle dt, \end{aligned}$$

for some  $z_1, z_2 \in \Lambda$ .

We now apply (25) with

$$g(x, t) = e^{ik \cdot x} e^{ilt} = \cos(k \cdot x + lt) + i \sin(k \cdot x + lt), \quad k \in \frac{2\pi}{2N} \Lambda, l \in \frac{2\pi}{\beta} \mathbb{Z}.$$

Then  $g$  satisfies the assumptions above, with  $q = 2/3$ . Since  $a'(z_1, t)^2 + b'(z_2, t)^2 \leq 2l^2$ , the first term on the right-hand-side in (25) is at most

$$\frac{8l^2\mu|D|}{2\delta^2/3} \leq \frac{24\beta|\Lambda|}{\delta}l^2.$$

Here we used also (22). Next,

$$\begin{aligned} (Lg(\cdot, t))(x) &= \sum_{y \in \Lambda} L(x, y)g(y, t) = \sum_{y \in \Lambda} L(0, y - x)e^{ik \cdot (y-x)}e^{ik \cdot x}e^{ilt} \\ &= \hat{L}(k)g(x, t), \end{aligned}$$

so the second term on the right-hand-side of (25) is

$$2\lambda \int_0^\beta \langle Lg(\cdot, t), \overline{g(\cdot, t)} \rangle dt = 2\lambda \hat{L}(k)\beta|\Lambda|.$$

In the left-hand-side of (25) we have

$$2\lambda \int_0^\beta \langle Lg(\cdot, t), \sigma(\cdot, t) \rangle dt = 2\lambda \hat{L}(k) \sum_{x \in \Lambda} \int_0^\beta \sigma(x, t)e^{ik \cdot x}e^{ilt} dt,$$

and

$$-\frac{1}{2\delta} \sum_{x \in \Lambda} \int_0^\beta g''(x, t)\sigma(x, t) dt = \frac{l^2}{2\delta} \sum_{x \in \Lambda} \int_0^\beta \sigma(x, t)e^{ik \cdot x}e^{ilt} dt.$$

Using the relation  $|z|^2 = z\bar{z}$  and the translation-invariance of  $\mu$ , it follows that the left-hand-side of (25) equals

$$\left(2\lambda \hat{L}(k) + \frac{l^2}{2\delta}\right)^2 \beta|\Lambda|\hat{c}(k, l).$$

Putting it all together gives

$$(26) \quad \hat{c}(k, l) \leq \frac{2\lambda \hat{L}(k) + 48l^2/2\delta}{(2\lambda \hat{L}(k) + l^2/2\delta)^2} \leq \frac{48}{2\lambda \hat{L}(k) + l^2/2\delta},$$

as required.  $\square$

**2.2. Proof of Lemma 2.2.** Let  $\tau : \Lambda \rightarrow \Lambda$  be any function. We extend  $\tau$  to a function  $\Lambda \times [0, \beta]^{\mathfrak{p}} \rightarrow \Lambda \times [0, \beta]^{\mathfrak{p}}$ , which we also denote by  $\tau$ , by letting  $\tau(x, t) = (\tau(x), t)$ . Let  $\mathbf{h} = (h(x, t) : x \in \Lambda, t \in [0, \beta]^{\mathfrak{p}})$ . We write  $h \circ \tau$  for the usual composition of the functions  $h$  and  $\tau$ , given by  $(h \circ \tau)(x, t) = h(\tau(x, t)) = h(\tau(x), t)$ . We also write  $\mathbf{h} \circ \tau = ((h \circ \tau)(x, t) : x \in \Lambda, t \in [0, \beta]^{\mathfrak{p}})$ . Note that if  $\tau_1, \tau_2 : \Lambda \rightarrow \Lambda$  then  $(\mathbf{h} \circ \tau_1) \circ \tau_2 = \mathbf{h} \circ (\tau_1 \circ \tau_2)$ , by the usual associativity of function composition.

A function  $\alpha : \Lambda \rightarrow \Lambda$  is an *automorphism* if, firstly, it is a bijection, and, secondly,  $x \sim y$  if and only if  $\alpha(x) \sim \alpha(y)$  for all  $x, y \in \Lambda$ . Since  $Z(\mathbf{h})$  only depends on  $\Lambda$  through its connectivity structure, we see that

$$(27) \quad Z(\mathbf{h} \circ \alpha) = Z(\mathbf{h}) \text{ for all automorphisms } \alpha : \Lambda \rightarrow \Lambda.$$

For any  $y \in \Lambda$  the map  $\alpha : x \mapsto x + y$  is an automorphism (addition is coordinate-wise and interpreted modulo  $2N$ ). Also, any permutation of the coordinates  $(x_1, \dots, x_d)$  is an automorphism.

The following functions  $\rho, \rho^+, \rho^- : \Lambda \rightarrow \Lambda$  will be particularly important in our proof of Lemma 2.2. We let  $\rho$  be the automorphism of  $\Lambda$  given by:

$$\rho : (x_1, x_2, \dots, x_d) \mapsto (2N - 1 - x_1, x_2, \dots, x_d).$$

We may think of  $\rho$  geometrically as a ‘reflection’ in a plane parallel to, and ‘just to the left of’, the first coordinate plane. Writing

$$\Lambda^+ = \{(x_1, \dots, x_d) \in \Lambda : 0 \leq x_1 \leq N - 1\}, \text{ and}$$

$$\Lambda^- = \{(x_1, \dots, x_d) \in \Lambda : N \leq x_1 \leq 2N - 1\},$$

it follows that  $\rho$  bijectively maps  $\Lambda^+$  to  $\Lambda^-$  and  $\Lambda^-$  to  $\Lambda^+$ . Next we define the functions  $\rho^+, \rho^- : \Lambda \rightarrow \Lambda$  by

$$\rho^+(x) = \begin{cases} x, & \text{if } x \in \Lambda^+, \\ \rho(x), & \text{if } x \in \Lambda^-, \end{cases} \quad \text{and} \quad \rho^-(x) = \begin{cases} x, & \text{if } x \in \Lambda^-, \\ \rho(x), & \text{if } x \in \Lambda^+. \end{cases}$$

Note that  $\rho^+$  and  $\rho^-$  are *not* bijections, and in particular not automorphisms.

For  $\mathbf{h} : \Lambda \times [0, \beta]^p \rightarrow \mathbb{R}$  we define the following two numbers:

- (1)  $N(\mathbf{h})$  is the number of unordered pairs of adjacent elements  $x \sim y$  of  $\Lambda$  such that  $h(x, \cdot) \neq h(y, \cdot)$ ;
- (2)  $N^\pm(\mathbf{h})$  is the number of pairs of adjacent elements  $x \sim y$  such that  $x \in \Lambda^+, y \in \Lambda^-$ , and  $h(x, \cdot) \neq h(y, \cdot)$ .

Equality of functions may here be interpreted pointwise or in the almost-everywhere sense, this makes no difference to our results. Lemma 2.2 follows from the following result:

LEMMA 2.4. *For any bounded, twice differentiable  $\mathbf{h} : \Lambda \times [0, \beta]^p \rightarrow \mathbb{R}$  we have that*

- (1)  $Z(\mathbf{h})^2 \leq Z(\mathbf{h} \circ \rho^+)Z(\mathbf{h} \circ \rho^-)$ , and
- (2)  $N(\mathbf{h}) \geq \min\{N(\mathbf{h} \circ \rho^+), N(\mathbf{h} \circ \rho^-)\}$ , this inequality being strict if  $N^\pm(\mathbf{h}) > 0$ .

Before proving Lemma 2.4 we show how it implies Lemma 2.2.

*Proof of Lemma 2.2.* The set of all functions  $\tau : \Lambda \rightarrow \Lambda$  is finite, so  $Z(\mathbf{h} \circ \tau)$  attains its maximum over  $\tau$ . Let  $\tau$  be chosen so that

- (1)  $Z(\mathbf{h} \circ \tau)$  is maximal, and
- (2)  $N(\mathbf{h} \circ \tau)$  is minimal among the maximizers of  $Z(\mathbf{h} \circ \tau)$ .

Suppose  $N(\mathbf{h} \circ \tau) > 0$ . By automorphism invariance (27) we may then assume that  $N^\pm(\mathbf{h} \circ \tau) > 0$ . It then follows from the second part of Lemma 2.4 that

$$(28) \quad N(\mathbf{h} \circ \tau) > \min\{N(\mathbf{h} \circ (\tau \circ \rho^+)), N(\mathbf{h} \circ (\tau \circ \rho^-))\}.$$

By our choice of  $\tau$ , both  $Z(\mathbf{h} \circ (\tau \circ \rho^+))$  and  $Z(\mathbf{h} \circ (\tau \circ \rho^-))$  are at most equal to  $Z(\mathbf{h} \circ \tau)$ , and in light also of (28) one of them must be strictly smaller than  $Z(\mathbf{h} \circ \tau)$ . This is a contradiction, however, since the first part of Lemma 2.4 would then give  $Z(\mathbf{h} \circ \tau)^2 < Z(\mathbf{h} \circ \tau)^2$ .

It follows that  $N(\mathbf{h} \circ \tau) = 0$ , which is to say that  $h(\tau(x), \cdot) = h(\tau(y), \cdot)$  for all  $x \sim y$ , and hence (since  $\Lambda$  is connected) for all  $x, y \in \Lambda$ . The result follows.  $\square$

To prove Lemma 2.4 we need one more preliminary result. We let  $\Sigma^+$  denote the set of functions  $\Lambda^+ \times [0, \beta]^{\mathfrak{p}} \rightarrow \{-1, +1\}$  which are right continuous, and have left limits, in the second argument. We let  $\mathcal{F}^+$  denote the natural sigma-field on  $\Sigma^+$ , generated by finite-dimensional projections, as in (7). We let  $\mathcal{B}$  denote the Borel sigma-field on  $[0, \beta]^{\mathfrak{p}}$ . For  $\sigma \in \Sigma$  and  $A : \Sigma^+ \rightarrow \mathbb{R}$  we use the shorthand  $A(\sigma)$  for  $A$  applied to the restriction of  $\sigma$  to  $\Lambda^+$ . For  $\tau : \Lambda \rightarrow \Lambda$  we write  $A \circ \tau$  for the function  $\sigma \mapsto A(\sigma \circ \tau)$ , where  $\sigma \circ \tau$  is as defined in the paragraph after Lemma 2.2. To simplify the notation we write  $A^+$  for  $A \circ \rho^+$  and  $A^-$  for  $A \circ \rho^-$ . Note that  $A^+$  and  $A^-$  have domain  $\Sigma$  rather than  $\Sigma^+$ . Recall the measure  $E$  from Section 1.1.

LEMMA 2.5. *Let  $J$  be a finite set. Let  $A, B : \Sigma^+ \rightarrow \mathbb{R}$  be bounded and  $\mathcal{F}^+$ -measurable, and for each  $j \in J$  let  $C_j, D_j : [0, \beta]^{\mathfrak{p}} \times \Sigma^+ \rightarrow \mathbb{R}$  be bounded and  $\mathcal{B} \times \mathcal{F}^+$ -measurable. Then*

$$\begin{aligned}
 (29) \quad E & \left[ \exp \left( A^+(\sigma) + B^-(\sigma) + \sum_{j \in J} \int_0^\beta C_{j,t}^+(\sigma) D_{j,t}^-(\sigma) dt \right) \right]^2 \\
 & \leq E \left[ \exp \left( A^+(\sigma) + A^-(\sigma) + \sum_{j \in J} \int_0^\beta C_{j,t}^+(\sigma) C_{j,t}^-(\sigma) dt \right) \right] \\
 & \quad \cdot E \left[ \exp \left( B^+(\sigma) + B^-(\sigma) + \sum_{j \in J} \int_0^\beta D_{j,t}^+(\sigma) D_{j,t}^-(\sigma) dt \right) \right]
 \end{aligned}$$

*Proof.* We first make the following observation. Let  $F, G : \Sigma^+ \rightarrow \mathbb{R}$  be bounded and  $\mathcal{F}^+$ -measurable. Then

$$\begin{aligned}
 (30) \quad E[F^+(\sigma)G^-(\sigma)]^2 &= E[F(\rho^+(\sigma))G(\rho^-(\sigma))]^2 \\
 &= E[F(\rho^+(\sigma))F(\rho^-(\sigma))]E[G(\rho^+(\sigma))G(\rho^-(\sigma))] \\
 &= E[F^+(\sigma)F^-(\sigma)]E[G^+(\sigma)G^-(\sigma)].
 \end{aligned}$$

This is because the sets  $\Lambda^+ = \rho^+(\Lambda^+)$  and  $\Lambda^- = \rho^-(\Lambda^+)$  are disjoint, and the random variables  $\rho^+(\sigma)$  and  $\rho^-(\sigma)$  therefore independent and identically distributed under  $E$ . For the same reason,

$$\begin{aligned}
 (31) \quad E[F^+(\sigma)F^-(\sigma)] &= E[F(\sigma)]^2 \geq 0, \text{ and} \\
 E[G^+(\sigma)G^-(\sigma)] &= E[G(\sigma)]^2 \geq 0.
 \end{aligned}$$

Turning now to (29), the integrand in the left-hand-side may be written as the sum

$$(32) \quad \sum_{k \geq 0} \frac{1}{k!} \exp(A^+(\sigma) + B^-(\sigma)) \left( \sum_{j \in J} \int_0^\beta C_{j,t}^+(\sigma) D_{j,t}^-(\sigma) dt \right)^k.$$

Taking the expectation inside the sum and expanding the last factor, each summand in (32) may in turn be written as a sum over  $j_1, \dots, j_k \in J$  of a repeated integral over  $t_1, \dots, t_k \in [0, \beta]^{\mathfrak{p}}$  of a term of the form

$$(33) \quad \frac{1}{k!} E[e^{A^+(\sigma)} C_{j_1, t_1}^+(\sigma) \cdots C_{j_k, t_k}^+(\sigma) \cdot e^{B^-(\sigma)} D_{j_1, t_1}^-(\sigma) \cdots D_{j_k, t_k}^-(\sigma)].$$

The latter expectation is of the form  $E[F^+(\sigma)G^-(\sigma)]$ , with

$$\begin{aligned} F(\sigma) &= e^{A(\sigma)} C_{j_1, t_1}(\sigma) \cdots C_{j_k, t_k}(\sigma), \text{ and} \\ G(\sigma) &= e^{B(\sigma)} D_{j_1, t_1}(\sigma) \cdots D_{j_k, t_k}(\sigma). \end{aligned}$$

Using (30), and slightly abusing notation, we therefore see that the left-hand-side of (29) equals

$$\begin{aligned} & \left( \sum_{k \geq 0} \sum_{j_1, \dots, j_k \in J} \int_0^\beta dt_1 \cdots \int_0^\beta dt_k \frac{1}{k!} \sqrt{E[F^+(\sigma)F^-(\sigma)]} \sqrt{E[G^+(\sigma)G^-(\sigma)]} \right)^2 \\ & \leq \sum_{k \geq 0} \frac{1}{k!} \sum_{j_1, \dots, j_k \in J} \int_0^\beta dt_1 \cdots \int_0^\beta dt_k E[F^+(\sigma)F^-(\sigma)] \\ & \quad \cdot \sum_{k \geq 0} \frac{1}{k!} \sum_{j_1, \dots, j_k \in J} \int_0^\beta dt_1 \cdots \int_0^\beta dt_k E[G^+(\sigma)G^-(\sigma)]. \end{aligned}$$

Here we used the Cauchy–Schwarz inequality as well as (31). Reversing the steps leading up to (33) for each of the two factors gives the result.  $\square$

*Proof of Lemma 2.4.* For the first part, the aim is to write  $Z(\mathbf{h})$  (see (19)) in terms of suitably chosen  $A$ ,  $B$ ,  $C_{j,t}$  and  $D_{j,t}$ , and then apply Lemma 2.5. To begin with, for simplicity of notation, fix  $t \in [0, \beta]^{\mathfrak{p}}$  and write  $\sigma(x)$  and  $h(x)$  for  $\sigma(x, t)$  and  $h(x, t)$ , respectively. We have by (17) that

$$\langle L(\sigma + h), \sigma + h \rangle = \sum_{x \sim y} (\sigma(x) + h(x) - \sigma(y) - h(y))^2,$$

which splits into the three sums

$$(34) \quad \sum_{\substack{x \sim y \\ x, y \in \Lambda^+}} (\sigma(x) + h(x) - \sigma(y) - h(y))^2,$$

$$(35) \quad \sum_{\substack{x \sim y \\ x, y \in \Lambda^-}} (\sigma(x) + h(x) - \sigma(y) - h(y))^2,$$



and

$$(36) \quad \sum_{\substack{x \sim y \\ x \in \Lambda^+, y \in \Lambda^-}} (\sigma(x) + h(x) - \sigma(y) - h(y))^2.$$

Write  $x \sim \Lambda^-$  (respectively,  $x \sim \Lambda^+$ ) to denote that  $x$  is adjacent to some element of  $\Lambda^-$  (respectively,  $\Lambda^+$ ). Then (36) equals

$$(37) \quad \begin{aligned} & \sum_{\substack{x \in \Lambda^+ \\ x \sim \Lambda^-}} (\sigma(x) + h(x))^2 + \sum_{\substack{y \in \Lambda^- \\ y \sim \Lambda^+}} (\sigma(y) + h(y))^2 \\ & - \sum_{\substack{x \sim y \\ x \in \Lambda^+, y \in \Lambda^-}} 2(\sigma(x) + h(x))(\sigma(y) + h(y)). \end{aligned}$$

Since  $\rho^-$  is a bijection from  $\Lambda^+$  to  $\Lambda^-$ , (35) equals

$$\sum_{\substack{x \sim y \\ x, y \in \Lambda^+}} ((\sigma \circ \rho^-)(x) + (h \circ \rho^-)(x) - (\sigma \circ \rho^-)(y) - (h \circ \rho^-)(y))^2,$$

and the second sum in (37) equals

$$\sum_{\substack{y \in \Lambda^+ \\ y \sim \Lambda^-}} ((\sigma \circ \rho^-)(y) + (h \circ \rho^-)(y))^2$$

Moreover, if  $x \in \Lambda^+$ ,  $y \in \Lambda^-$  with  $x \sim y$ , then  $y = \rho^-(x)$ . So the last term in (37) equals

$$\sum_{\substack{x \in \Lambda^+ \\ x \sim \Lambda^-}} 2(\sigma(x) + h(x))((\sigma \circ \rho^-)(x) + (h \circ \rho^-)(x)).$$

Recall that  $\rho^+(x) = x$  for  $x \in \Lambda^+$ . Reintroducing  $t$  into our notation, it follows that

$$(38) \quad \begin{aligned} & -\lambda \int_0^\beta \langle L[\sigma(\cdot, t) + h(\cdot, t)], [\sigma(\cdot, t) + h(\cdot, t)] \rangle dt = \\ & = A_1^+(\sigma) + B_1^-(\sigma) + \sum_{\substack{x \in \Lambda^+ \\ x \sim \Lambda^-}} \int_0^\beta C_{x,t}^+(\sigma) D_{x,t}^-(\sigma) dt, \end{aligned}$$

where  $A_1, B_1, C_{x,t}, D_{x,t} : \Sigma \rightarrow \mathbb{R}$  are given by

$$\begin{aligned} A_1(\sigma) = & -\lambda \int_0^\beta \left( \sum_{\substack{x \in \Lambda^+ \\ x \sim \Lambda^-}} (\sigma(x, t) + h(x, t))^2 + \right. \\ & \left. + \sum_{\substack{x \sim y \\ x, y \in \Lambda^+}} (\sigma(x, t) + h(x, t) - \sigma(y, t) - h(y, t))^2 \right) dt, \end{aligned}$$

$$\begin{aligned}
B_1(\sigma) = & -\lambda \int_0^\beta \left( \sum_{\substack{x \in \Lambda^+ \\ x \sim \Lambda^-}} (\sigma(x, t) + (h \circ \rho^-)(x, t))^2 + \right. \\
& \left. + \sum_{\substack{x \sim y \\ x, y \in \Lambda^+}} (\sigma(x, t) + (h \circ \rho^-)(x, t) - \sigma(y, t) - (h \circ \rho^-)(y, t))^2 \right) dt,
\end{aligned}$$

and

$$\begin{aligned}
(39) \quad C_{x,t}(\sigma) &= \sqrt{2\lambda}(\sigma(x, t) + h(x, t)), \\
D_{x,t}(\sigma) &= \sqrt{2\lambda}(\sigma(x, t) + (h \circ \rho^-)(x, t)).
\end{aligned}$$

Next,

$$(40) \quad \frac{1}{2\delta} \sum_{x \in \Lambda} \int_0^\beta h''(x, t) \sigma(x, t) dt = A_2^+(\sigma) + B_2^-(\sigma),$$

where

$$A_2(\sigma) = \frac{1}{2\delta} \sum_{x \in \Lambda^+} \int_0^\beta h''(x, t) \sigma(x, t) dt$$

and

$$B_2(\sigma) = \frac{1}{2\delta} \sum_{x \in \Lambda^+} \int_0^\beta (h \circ \rho^-)''(x, t) \sigma(x, t) dt.$$

Thus, from (19),  $Z(\mathbf{h})$  is of the form of the left-hand-side of (29), with  $A(\sigma) = A_1(\sigma) + A_2(\sigma)$ ,  $B(\sigma) = B_1(\sigma) + B_2(\sigma)$ ,  $J = \{x \in \Lambda^+ : x \sim \Lambda^-\}$ , and  $C_{x,t}$  and  $D_{x,t}$  as in (39).

Reversing the steps leading to (38) and (40) (and recalling that  $\rho^+$  is a bijection from  $\Lambda^-$  to  $\Lambda^+$ ) shows that

$$Z(\mathbf{h} \circ \rho^+) = E \left[ \exp \left( A^+(\sigma) + A^-(\sigma) + \sum_{j \in J} \int_0^\beta C_{j,t}^+(\sigma) C_{j,t}^-(\sigma) dt \right) \right]$$

and that

$$Z(\mathbf{h} \circ \rho^-) = E \left[ \exp \left( B^+(\sigma) + B^-(\sigma) + \sum_{j \in J} \int_0^\beta D_{j,t}^+(\sigma) D_{j,t}^-(\sigma) dt \right) \right].$$

The first part of the lemma now follows from Lemma 2.5.

For the second part define the numbers

- $N^+(\mathbf{h})$  as the number of unordered pairs of adjacent elements  $x \sim y$  of  $\Lambda^+$  such that  $h(x, \cdot) \neq h(y, \cdot)$ ; and
- $N^-(\mathbf{h})$  as the number of unordered pairs of adjacent elements  $x \sim y$  of  $\Lambda^-$  such that  $h(x, \cdot) \neq h(y, \cdot)$ .

Then  $N(\mathbf{h}) = N^+(\mathbf{h}) + N^-(\mathbf{h}) + N^\pm(\mathbf{h})$ , whereas  $N(\mathbf{h} \circ \rho^+) = 2N^+(\mathbf{h})$  and  $N(\mathbf{h} \circ \rho^-) = 2N^-(\mathbf{h})$ . The result follows immediately from this observation.  $\square$

**2.3. Proof of Lemma 2.3.** Write  $D_x = \{t_1^x, t_2^x, \dots, t_{|D_x|}^x\}$  for the points of  $D_x$  ordered so that  $0 < t_1^x < t_2^x < \dots < t_{|D_x|}^x < \beta$ . For convenience we also write  $t_{|D_x|+1}^x = t_1^x$ . Note that if  $h$  is twice differentiable, then

$$\begin{aligned}
 \int_0^\beta h''(t) \sigma(x, t) dt &= \sum_{j=1}^{|D_x|} \int_{t_j^x}^{t_{j+1}^x} h''(t) \sigma(x, t) dt \\
 (41) \qquad &= \sum_{j=1}^{|D_x|} \sigma(x, t_j^x) (h'(t_{j+1}^x) - h'(t_j^x)) \\
 &= -2 \sum_{j=1}^{|D_x|} h'(t_j^x) \sigma(x, t_j^x).
 \end{aligned}$$

Here we used the fact that  $\sigma(x, t_j^x) = -\sigma(x, t_{j+1}^x)$ . Moreover, since  $\sigma(x, t) = (-1)^{\xi_x + |D_x \cap [0, t]|}$  we have that  $\sigma(x, t_j^x) = (-1)^{\xi_x + j}$ . Hence

$$(42) \qquad Z(h) = E \left[ \exp \left( -\lambda \int_0^\beta \langle L\sigma(\cdot, t), \sigma(\cdot, t) \rangle dt - \frac{1}{\delta} \sum_{x \in \Lambda} \sum_{j=1}^{|D_x|} h'(t_j^x) (-1)^{\xi_x + j} \right) \right].$$

Note that the form (42) does not require  $h$  to be twice differentiable. We say that  $h$  is *weakly differentiable*, and that  $h'$  is a *weak derivative* of  $h$ , if there is a constant  $c$  such that

$$h(t) = \int_0^t h'(s) ds + c \quad \text{for all } 0 \leq t < \beta.$$

Since weak derivatives are defined up to a set of zero measure, (42) is well-defined if we take  $h'$  to be any weak derivative of  $h$ . In this section we will let  $Z(h)$  denote the quantity in (42), and the standing assumption on  $h$  will be that it is weakly differentiable with a bounded weak derivative. Also note that  $Z(h) = Z(h + c)$  for any constant  $c$ , so we may occasionally assume that  $h(0) = 0$ .

By the monotonicity of  $\zeta$ , Lemma 2.3 will be proved if we show:

$$(43) \qquad \text{there is } 0 \leq r \leq \|h'\|_\infty \text{ such that } Z(h)/Z(0) \leq \zeta(r).$$

The proof of (43) is preceded by a number of preliminary results, of which the following general fact is the first. For each  $n \geq 1$  let

$$\mathcal{O}^{(n)} = \bigcup_{k=0}^{2^{n-1}-1} [(2k+1)2^{-n}, (2k+2)2^{-n})$$

be the union of those level  $n$  dyadic subintervals of  $[0, 1)$  whose left endpoints are odd multiples of  $2^{-n}$ . Write  $B_n(t) = \mathbb{1}\{t \in \mathcal{O}^{(n)}\}$  for  $t \in [0, 1]$ .

LEMMA 2.6. *Let  $m \geq 1$  and let  $T = (T_1, \dots, T_m) \in [0, 1]^m$  be a random vector with square integrable density  $p : [0, 1]^m \rightarrow [0, \infty)$ . Then for each  $b \in \{0, 1\}^m$ ,*

$$P((B_n(T_1), \dots, B_n(T_m)) = b) \rightarrow \frac{1}{2^m}, \quad \text{as } n \rightarrow \infty.$$

Intuitively, Lemma 2.6 states that the level  $n$  binary digits of the  $T_j$  are asymptotically independent and uniform as  $n \rightarrow \infty$ .

*Proof.* For each  $n \geq 1$  and  $A \subseteq \{1, \dots, m\}$ , let

$$R_n^A(t) = \prod_{j \in A} (-1)^{B_n(t_j)}, \quad t \in [0, 1]^m,$$

denote the Rademacher function. Then  $(R_n^A : A \neq \emptyset, n \geq 1)$  are orthonormal in the Hilbert space  $L^2([0, 1]^m)$ . Writing

$$\hat{p}(A, n) = E(R_n^A(T)) = \int_0^1 \cdots \int_0^1 p(t_1, \dots, t_m) R_n^A(t_1, \dots, t_m) dt_1 \dots dt_m,$$

it follows (from Bessel's inequality or otherwise) that for each  $\emptyset \neq A \subseteq \{1, \dots, m\}$  we have  $\sum_{n \geq 1} \hat{p}(A, n)^2 < \infty$ . In particular,  $\hat{p}(A, n) \rightarrow 0$  as  $n \rightarrow \infty$ . For any  $A \subseteq \{1, \dots, m\}$ ,

$$\begin{aligned} \mathbb{I}\{B_n(T_j) = 1 \ \forall j \in A, \ B_n(T_j) = 0 \ \forall j \notin A\} &= \\ &= 2^{-m} \prod_{j \in A} (1 - (-1)^{B_n(T_j)}) \prod_{j \notin A} (1 + (-1)^{B_n(T_j)}). \end{aligned}$$

Expanding the products on the right, we obtain a sum of terms of the form  $\pm R_n^C(T)$  for  $C \subseteq \{1, \dots, m\}$ . The term  $R_n^\emptyset(T) = 1$  appears exactly once. Taking expected value and letting  $n \rightarrow \infty$  gives the result.  $\square$

The following technical lemma will enable us to apply Lemma 2.6 to the process  $D$ .

LEMMA 2.7. *For each  $x \in \Lambda$ , let  $m_x \geq 0$  be an integer, and for  $j \in \{1, \dots, 2m_x\}$  let  $I_j^x = [a_j^x, b_j^x] \subseteq [0, \beta)$  be intervals such that  $0 < a_1^x < b_1^x \leq a_2^x < b_2^x \leq \dots \leq a_{2m_x}^x < b_{2m_x}^x < \beta$ . Let  $A$  denote the event that: for each  $x \in \Lambda$  we have  $|D_x| = 2m_x$ , and for all  $j \in \{1, \dots, 2m_x\}$  we have  $t_j^x \in I_j^x$ . Then the law of  $(t_j^x : x \in \Lambda, j \in \{1, \dots, 2m_x\})$  under  $\mu(\cdot \mid A)$  has a square-integrable density with respect to Lebesgue measure on  $I := \prod_{x \in \Lambda} \prod_{j=1}^{2m_x} I_j^x$ .*

*Proof.* Recall that  $E = E_0 \times E_\times$ , where  $E_0$  governs  $\xi$  and  $E_\times$  governs  $D$ . The law of  $D$  under  $\mu$  has density

$$(44) \quad q(D) = E_0 \left[ \frac{1}{Z_\Lambda} \exp \left( \lambda \sum_{x \sim y} \int_0^\beta \sigma(x, t) \sigma(y, t) dt \right) \right]$$

with respect to  $E_\times$ . From standard properties of conditional expectation it follows that the law of  $D$  under  $\mu(\cdot | A)$  has density

$$\tilde{q}(D) = \frac{q(D)}{E_\times[q(D) | A]}$$

with respect to  $E_\times(\cdot | A)$ . The law of  $(t_j^x : x \in \Lambda, j \in \{1, \dots, 2m_x\})$  under  $E_\times(\cdot | A)$  is the uniform distribution on  $I$ , by standard properties of Poisson processes. It is clear from (44) that  $\tilde{q}$  is square integrable.  $\square$

Let  $r \in \mathbb{R}$  be a real number and  $n \geq 1$  an integer, and let

$$W'_{r,n}(t) = r(-1)^{\lfloor 2^n t / \beta \rfloor}.$$

Thus  $W'_{r,n}(t)$  takes the two values  $\pm r$ , and changes sign at the level  $n$  dyadics, ie points of the form  $t = k2^{-n}\beta$  for  $k \in \{0, 1, \dots, 2^n - 1\}$ . Let  $W_{r,n}$  be the antiderivative of  $W'_{r,n}$ , given by

$$W_{r,n}(t) = \int_0^t W'_{r,n}(s) ds, \quad \text{for all } 0 \leq t < \beta.$$

See Figure 1. Recall that  $\zeta(r) = \mu[\cosh(r/\delta)^{|D|}]$ .

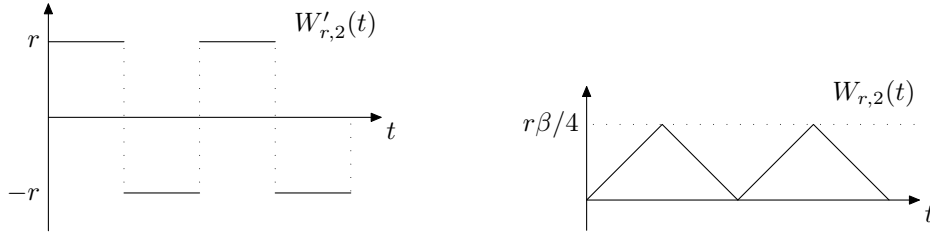


FIGURE 1. The functions  $W'_{r,n}(t)$  (left) and  $W_{r,n}(t)$  (right) for  $n = 2$ .

LEMMA 2.8. *We have that*

$$\lim_{n \rightarrow \infty} Z(W_{r,n}) = \zeta(r)Z(0).$$

*Proof.* From (42) and rotational invariance we need to show that

$$(45) \quad \frac{Z(W_{r,n})}{Z(0)} = \mu \left[ \exp \left( \frac{r}{\delta} \sum_{x \in \Lambda} \sum_{j=1}^{|D_x|} (-1)^{\xi_x + j + \lfloor 2^n t_j^x / \beta \rfloor} \right) \right] \rightarrow \zeta(r)$$

as  $n \rightarrow \infty$ . To motivate the argument that follows, let  $(Y_j^x : x \in \Lambda, j \geq 1)$  be independent random variables, each taking the values  $\pm 1$  with equal probability under  $\mu$ . Then (by conditioning on the  $|D_x|$ )

$$\mu \left[ \exp \left( \frac{r}{\delta} \sum_{x \in \Lambda} \sum_{j=1}^{|D_x|} Y_j^x \right) \right] = \mu [\mu[\exp(rY_1^0/\delta)]^{|D|}] = \mu[\cosh(r/\delta)^{|D|}] = \zeta(r).$$

The strategy for proving (45) will be to first condition on the ‘rough’ locations of the  $t_j^x$ , and then use Lemma 2.6 to deduce that the conditional joint distribution of the numbers  $(-1)^{\xi_x+j+\lfloor 2^n t_j^x/\beta \rfloor}$  approaches that of the  $Y_j^x$ . Here are the details.

Writing

$$H_n = \frac{1}{\delta} \sum_{x \in \Lambda} \sum_{j=1}^{|D_x|} (-1)^{\xi_x+j+\lfloor 2^n t_j^x/\beta \rfloor},$$

first note that the sequence  $(e^{rH_n} : n \geq 1)$  is uniformly integrable under  $\mu$ , for each  $r \in \mathbb{R}$ . Indeed, a sufficient condition for uniform integrability is that

$$\sup_{n \geq 1} \mu[|e^{rH_n}|^2] = \sup_{n \geq 1} \mu[e^{2rH_n}] < \infty.$$

This follows from Lemma 2.1 and the fact that  $|H_n| \leq |D|/\delta$ .

Let  $\varepsilon > 0$  be arbitrary. Let  $M$  be some (large) integer, and let  $A_M$  denote the event that each  $|D_x|$  is at most  $2M$ . Then  $\mu(A_M) \rightarrow 1$  as  $M \rightarrow \infty$ . Next, let  $L$  be another (large) integer, and let  $B_L$  denote the event that for each  $x \in \Lambda$  there are integers  $0 \leq k_1^x < k_2^x < \dots < k_{|D_x|}^x \leq 2^L - 1$  such that each  $t_j^x \in [k_j^x 2^{-L}\beta, (k_j^x + 1)2^{-L}\beta)$ . Then also  $\mu(B_L) \rightarrow 1$  as  $L \rightarrow \infty$ . For each  $\alpha > 0$  we have that

$$\sup_{n \geq 1} \mu[e^{rH_n} \mathbb{1}_{A_M^c \cup B_L^c}] \leq \sup_{n \geq 1} \mu[e^{rH_n} \mathbb{1}\{e^{rH_n} > \alpha\}] + \alpha \mu(A_M^c \cup B_L^c).$$

By uniform integrability, this can be made smaller than  $\varepsilon$  by first choosing  $\alpha$  large enough that  $\sup_{n \geq 1} \mu[e^{rH_n} \mathbb{1}\{e^{rH_n} > \alpha\}] < \varepsilon/2$  and then  $M, L$  large enough that  $\alpha \mu(A_M^c \cup B_L^c) < \varepsilon/2$ . In what follows we assume that  $M$  and  $L$  are fixed, and large enough that

$$(46) \quad \sup_{n \geq 1} \mu[e^{rH_n} \mathbb{1}_{A_M^c \cup B_L^c}] < \varepsilon.$$

Let  $\mu'$  denote  $\mu$  conditioned on the following:

- (1) that  $A_M$  and  $B_L$  both occur;
- (2) the vector  $\xi = (\xi_x : x \in \Lambda)$ ;
- (3) the sizes  $|D_x| = 2m_x$  for all  $x \in \Lambda$ ;
- (4) and the numbers  $0 \leq k_1^x < k_2^x < \dots < k_{2m_x}^x \leq 2^L - 1$  such that each  $t_j^x \in [k_j^x 2^{-L}\beta, (k_j^x + 1)2^{-L}\beta)$ .

Let  $m = \sum_{x \in \Lambda} m_x$  so that  $|D| = 2m$ , and let

$$T_j^x = \frac{t_j^x - k_j^x 2^{-L}\beta}{2^{-L}\beta}.$$

Then  $T = (T_j^x : x \in \Lambda, j \in \{1, \dots, 2m_x\})$  is a random vector in  $[0, 1]^{2m}$ . Moreover, by Lemma 2.7 the law of  $T$  under  $\mu'$  has square integrable density with respect to Lebesgue measure on  $[0, 1]^{2m}$ .

Let  $n \geq 1$  and write  $X_j^x = \mathbb{I}\{T_j^x \in \mathcal{O}^{(L+n)}\}$ . Note that  $X_j^x$  has the same parity as  $\lfloor 2^{L+n} t_j^x / \beta \rfloor$ . Let  $a = (a_j^x : x \in \Lambda, 1 \leq j \leq 2m_x) \in \{-1, 1\}^{2m}$  be arbitrary, and write

$$X' = ((-1)^{\xi_x + j + X_j^x} : x \in \Lambda, j \in \{1, \dots, 2m_x\}).$$

By Lemma 2.6 we have that  $\mu'(X' = a) \rightarrow 2^{-m}$  as  $n \rightarrow \infty$  for any  $a \in \{-1, +1\}^{2m}$ , and hence that

$$\mu' \left[ \exp \left( \frac{r}{\delta} \sum_{x \in \Lambda} \sum_{j=1}^{2m_x} (-1)^{\xi_x + j + X_j^x} \right) \right] \rightarrow \mu' \left[ \exp \left( \frac{r}{\delta} \sum_{x \in \Lambda} \sum_{j=1}^{2m_x} Y_j^x \right) \right] = \cosh(r/\delta)^{2m}.$$

Now, with some slight abuse of notation for conditional expectation,

$$\begin{aligned} \frac{Z(W_{r,L+n})}{Z(0)} &= \mu \left[ \exp \left( \frac{r}{\delta} \sum_{x \in \Lambda} \sum_{j=1}^{|D_x|} (-1)^{\xi_x + j + X_j^x} \right) \right] \\ &= \mu[e^{rH_{L+n}} \mathbb{I}_{A_M^c \cup B_L^c}] + \mu \left[ \mu' \left[ \exp \left( \frac{r}{\delta} \sum_{x \in \Lambda} \sum_{j=1}^{2m_x} (-1)^{\xi_x + j + X_j^x} \right) \right] \right]. \end{aligned}$$

The first term is smaller than  $\varepsilon$ , by (46). For fixed  $M$  and  $L$ , the outer expectation in the second term is over a finite set of possibilities, namely the possible values of the  $\xi_x$ , the  $|D_x|$  (each being at most  $2M$ ) and the  $k_j^x$ , as listed in the definition of  $\mu'$ . By making  $n$  sufficiently large, we may by (47) assume that the integrand differs from  $\cosh(r)^{2m}$  by at most  $\varepsilon$  for each such possibility. It then follows that

$$\left| \frac{Z(W_{r,L+n})}{Z(0)} - \mu[\cosh(r/\delta)^{|D|}] \right| < 2\varepsilon.$$

The result follows.  $\square$

The rough strategy for proving (43) will be to define a procedure by which  $h$  can be altered so that it more and more resembles  $W_{r,n}$  for some  $r$ , whilst increasing  $Z(h)$ . First we need a Cauchy–Schwarz-type inequality along the lines of Lemma 2.5.

Let  $\theta : [0, \beta)^{\mathfrak{p}} \rightarrow [0, \beta)^{\mathfrak{p}}$  be given by  $\theta(t) = \beta - t$ . We may think of  $\theta$  as a reflection of the circle  $[0, \beta)^{\mathfrak{p}} = \{e^{2\pi i t / \beta} : t \in [0, \beta)\}$  in the real line. Recall that the measure  $E_{\times}[\cdot]$  governs  $D$  only, which is a Poisson process conditioned on each  $D_x$  having even size. For each  $x \in \Lambda$ , let  $D_x^+ = D_x \cap (0, \beta/2)$  and  $D_x^- = D_x \cap (\beta/2, \beta)$ . Write  $D^+ = (D_x^+ : x \in \Lambda)$  and  $D^- = (D_x^- : x \in \Lambda)$ ; also write  $\theta D^- = \{\theta t : t \in D^-\} \subseteq (0, \beta/2)$ .

LEMMA 2.9. *The measure  $E_{\times}[\cdot]$  is ‘reflection positive’ in that for any bounded, measurable function  $F$  of  $D^+$  we have that*

$$(48) \quad E_{\times}[F(D^+)F(\theta D^-)] \geq 0.$$

Consequently we have for all bounded measurable  $F, G$  that

$$(49) \quad E_{\times}[F(D^+)G(\theta D^-)]^2 \leq E_{\times}[F(D^+)F(\theta D^-)]E_{\times}[G(D^+)G(\theta D^-)].$$

*Proof.* For (48), condition on the parity of each  $|D_x^+|$ . Note that  $|D_x^-|$  necessarily has the same parity as  $|D_x^+|$ . Given this parity,  $\theta D_x^-$  is independent of, and identically distributed as,  $D_x^+$ . This gives

$$E_\times[F(D^+)F(\theta D^-)] = \sum_{j \in \{0,1\}^\Lambda} E_\times[F(D^+) \mid \forall x \in \Lambda, |D_x| \equiv j_x \pmod{2}]^2,$$

and hence (48).

The Cauchy–Schwarz inequality (49) is a standard consequence of (48) seeing as  $D^+$  and  $\theta D^-$  are identically distributed: for any  $t \in \mathbb{R}$  we have that

$$\begin{aligned} 0 &\leq E_\times[(F(D^+) + tG(D^+))(F(\theta D^-) + tG(\theta D^-))] \\ &= E_\times[F(D^+)F(\theta D^-)] + 2tE_\times[F(D^+)G(\theta D^-)] + t^2E_\times[G(D^+)G(\theta D^-)]. \end{aligned}$$

So the discriminant

$$4E_\times[F(D^+)G(\theta D^-)]^2 - 4E_\times[F(D^+)F(\theta D^-)]E_\times[G(D^+)G(\theta D^-)] \leq 0,$$

which gives (49).  $\square$

For any  $f : [0, \beta]^p \rightarrow \mathbb{R}$  define  $f_+$  and  $f_-$  by

$$(50) \quad f_+(t) = \begin{cases} f(t), & \text{if } t \in [0, \beta/2), \\ f(\theta t), & \text{if } t \in [\beta/2, \beta), \end{cases}$$

and

$$(51) \quad f_-(t) = \begin{cases} f(\theta t), & \text{if } t \in [0, \beta/2), \\ f(t), & \text{if } t \in [\beta/2, \beta). \end{cases}$$

If  $f$  has a weak derivative  $f'$  then  $f_+$  and  $f_-$  have weak derivatives  $f'_+$  and  $f'_-$  satisfying

$$(52) \quad f'_+(t) = \begin{cases} f'(t), & \text{if } t \in (0, \beta/2), \\ -f'(\theta t), & \text{if } t \in (\beta/2, \beta), \end{cases}$$

and

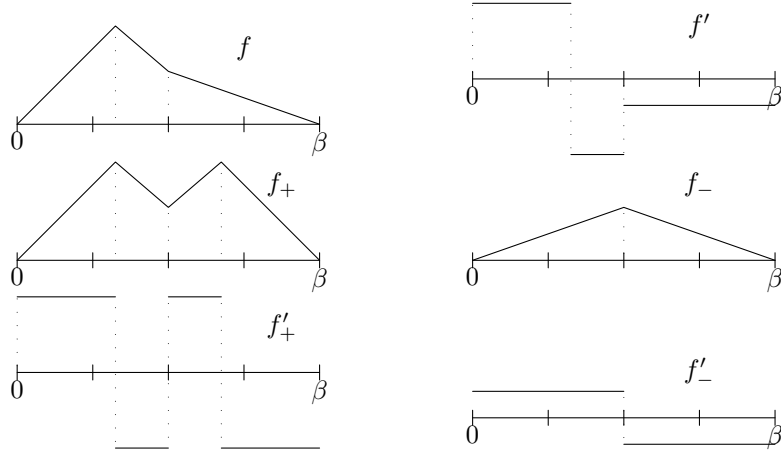
$$(53) \quad f'_-(t) = \begin{cases} -f'(\theta t), & \text{if } t \in (0, \beta/2), \\ f'(t), & \text{if } t \in (\beta/2, \beta). \end{cases}$$

See Figure 2 for an illustration.

The following result parallels Lemma 2.4.

**LEMMA 2.10.** *Let  $h : [0, \beta]^p \rightarrow \mathbb{R}$  have a bounded weak derivative. Then  $Z(h) \leq \max\{Z(h_+), Z(h_-)\}$ .*



FIGURE 2. An example of  $f_+$  and  $f_-$  and their derivatives.

*Proof.* Conditioning on  $\xi = (\xi_x : x \in \Lambda) \in \{0, 1\}^\Lambda$ , we may write  $Z(h) = \frac{1}{2^{|\Lambda|}} \sum_{\xi} Z(h \mid \xi)$ , where

$$(54) \quad Z(h \mid \xi) := E_{\times} \left[ \exp \left( -\lambda \int_0^\beta \langle L\sigma(\cdot, t), \sigma(\cdot, t) \rangle dt - \frac{1}{\delta} \sum_{x \in \Lambda} \sum_{j=1}^{|D_x|} (-1)^{j+\xi_x} h'(t_j^x) \right) \right].$$

We will express  $Z(h \mid \xi)$  in the form

$$(55) \quad Z(h \mid \xi) = E_{\times} [\exp (A_{\xi}(D^+) + B_{\xi}(\theta D^-))]$$

and then use Lemma 2.9. First we need some notation.

Write  $r_j^x$ ,  $j = 1, \dots, |D_x^+|$ , for the elements of  $D_x^+$  ordered so that  $r_j^x < r_{j+1}^x$  for all  $j$ . Also let  $r_0^x = 0$  and  $r_{|D_x^+|+1}^x = \beta/2$ . Similarly, write  $s_j^x$ ,  $j = 1, \dots, |D_x^-|$ , for the elements of  $D_x^-$  but ordered so that  $s_j^x > s_{j+1}^x$  for all  $j$ ; in other words so that  $\theta s_j^x < \theta s_{j+1}^x$  for all  $j$ . Also let  $s_0^x = 0$  and  $s_{|D_x^-|+1}^x = \beta/2$ . Note that

$$t_j^x = \begin{cases} r_j^x, & \text{for } 1 \leq j \leq |D_x^+|, \\ s_{|D_x^+|+j}^x, & \text{for } |D_x^+| + 1 \leq j \leq |D_x|. \end{cases}$$

See Figure 3.

In the last sum in (54) we have

$$\begin{aligned} \sum_{j=1}^{|D_x|} (-1)^{j+\xi_x} h'(t_j^x) &= \sum_{j=1}^{|D_x^+|} (-1)^{j+\xi_x} h'(r_j^x) + \sum_{j=1}^{|D_x^-|} (-1)^{|D_x^+|+j+\xi_x} h'(s_j^x) \\ &= \sum_{j=1}^{|D_x^+|} (-1)^{j+\xi_x} h'_+(r_j^x) + \sum_{j=1}^{|\theta D_x^-|} (-1)^{j+\xi_x} h'_-(\theta s_j^x). \end{aligned}$$

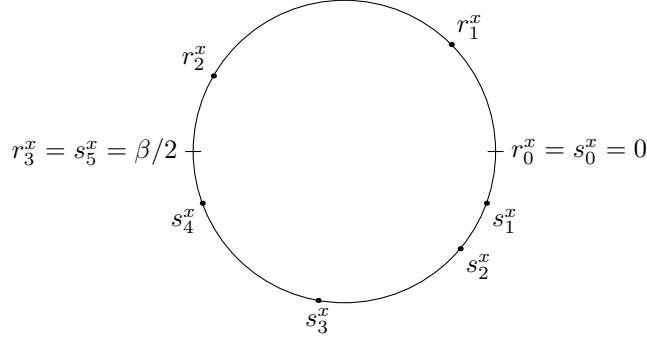


FIGURE 3. Ordering of the elements of  $D_x^+ = \{r_1^x, r_2^x\}$  and  $D_x^- = \{s_1^x, s_2^x, s_3^x, s_4^x\}$ .

Hence (55) holds with

$$A_\xi(D^+) = -\lambda \int_0^{\beta/2} \langle L\sigma(\cdot, t), \sigma(\cdot, t) \rangle dt - \frac{1}{\delta} \sum_{x \in \Lambda} \sum_{j=1}^{|D_x^+|} (-1)^{j+\xi_x} h'_+(r_j^x)$$

and

$$B_\xi(D^+) = -\lambda \int_0^{\beta/2} \langle L\sigma(\cdot, t), \sigma(\cdot, t) \rangle dt - \frac{1}{\delta} \sum_{x \in \Lambda} \sum_{j=1}^{|D_x^+|} (-1)^{j+\xi_x} h'_-(r_j^x).$$

Next note that

$$\begin{aligned} A_\xi(\theta D^-) &= \sum_{j=1}^{|\theta D_x^-|} (-1)^{j+\xi_x} h'_+(\theta s_j^x) = \sum_{j=1}^{|D_x^-|} (-1)^{j+\xi_x} (-h'_+(s_j^x)) \\ &= \sum_{j=1}^{|D_x^-|} (-1)^{j+\xi_x+1} h'_+(t_{|D_x^-|-j+1}^x) = \sum_{j=|D_x^+|+1}^{|D_x|} (-1)^{j+\xi_x} h'_+(t_j^x). \end{aligned}$$

Here we used the fact that  $|D_x^+| + |D_x^-| = |D_x|$  is even so that  $(-1)^{|D_x^+|} = (-1)^{|D_x^-|}$ . It follows that  $Z(h_+ | \xi) = E_\times [\exp(A_\xi(D^+) + A_\xi(\theta D^-))]$ . Similarly  $Z(h_- | \xi) = E_\times [\exp(B_\xi(D^+) + B_\xi(\theta D^-))]$ . Therefore we get from Lemma 2.9 that

$$\begin{aligned} Z(h | \xi)^2 &\leq E_\times [\exp(A_\xi(D^+) + A_\xi(\theta D^-))] \\ (56) \quad &\quad \cdot E_\times [\exp(B_\xi(D^+) + B_\xi(\theta D^-))] \\ &= Z(h_+ | \xi) Z(h_- | \xi). \end{aligned}$$

From (56) and the usual Cauchy–Schwarz inequality,

$$(57) \quad Z(h)^2 \leq \left( \frac{1}{2^{|\Lambda|}} \sum_{\xi \in \{0,1\}} \sqrt{Z(h_+ | \xi)} \sqrt{Z(h_- | \xi)} \right)^2 \leq Z(h_+) Z(h_-).$$

Finally, (57) implies that at least one of  $Z(h_+)$  and  $Z(h_-)$  is at least  $Z(h)$ , as required.  $\square$

DEFINITION 2.11 (Symmetrization). Let  $h : [0, \beta]^{\mathfrak{p}} \rightarrow \mathbb{R}$  have a bounded weak derivative and let  $t \in [0, \beta/2)$ . The symmetrization of  $h$  at  $t$  is the function  $g$  given by the completing the following steps:

- (1) Let  $\tilde{h} : s \mapsto h(s + t)$ ;
- (2) Let  $\tilde{h}_+$  and  $\tilde{h}_-$  be as in (50) and (51);
- (3) Let

$$\tilde{g} = \begin{cases} \tilde{h}_+, & \text{if } Z(\tilde{h}_+) \geq Z(\tilde{h}_-), \\ \tilde{h}_-, & \text{otherwise;} \end{cases}$$

- (4) Let  $g(s) = \tilde{g}(s - t)$ .

Note that if  $g$  is the symmetrization of  $h$  at  $t$  then  $g$  is symmetric at  $t$  and  $t + \beta/2$ ; also  $Z(g) \geq Z(h)$  by Lemma 2.10.

The strategy for proving (43) is to successively symmetrize  $h$  at dyadic points of finer and finer partition. Thereby our function more and more resembles  $W_{r,n}$  for some  $r$ . By Lemma 2.8 we know that  $\lim_{n \rightarrow \infty} Z(W_{r,n}) = \zeta(r)Z(0)$ . We now make this precise.

DEFINITION 2.12 (Snippet). Let  $f : [0, \beta]^{\mathfrak{p}} \rightarrow \mathbb{R}$ . For  $n \geq 0$  we call a function  $g : [0, \beta]^{\mathfrak{p}} \rightarrow \mathbb{R}$  a level  $n$  snippet of  $f$  if

- (1) there is  $k \in \{0, 1, \dots, 2^n - 1\}$  and  $a \in \{0, 1\}$  such that  $g(t) = f(k2^{-n}\beta + (-1)^a t)$  for all  $0 < t < 2^{-n}\beta$ , and
- (2)  $g(m2^{-n}\beta + t) = g(m2^{-n}\beta - t)$  for all  $m \in \{0, 1, \dots, 2^n - 1\}$  and all  $0 < t < 2^{-n}\beta$ .

Thus a level  $n$  snippet of  $f$  repeats the values that  $f$  takes on an interval  $(k2^{-n}\beta, (k+1)2^{-n}\beta)$ , but alternates between ‘the right way’ and ‘backwards’. Note that  $h_+$  and  $h_-$  are level 1 snippets of  $h$ .

LEMMA 2.13. Let  $h : [0, \beta]^{\mathfrak{p}} \rightarrow \mathbb{R}$  have a bounded weak derivative. There is a sequence  $(h_n : n \geq 0)$  of functions  $h_n : [0, \beta]^{\mathfrak{p}} \rightarrow \mathbb{R}$  such that

- (1)  $h_0 = h$ ,
- (2) for each  $n \geq 1$ ,  $h_n$  is a level  $n$  snippet of  $h_{n-1}$ , and
- (3) for each  $n \geq 1$ ,  $Z(h_n) \geq Z(h_{n-1})$ .

*Proof.* We are free to set  $h_0 = h$ . Let  $h_1$  be the symmetrization of  $h_0$  at 0, that is to say  $h_1$  is either  $h_+$  or  $h_-$ , chosen so that  $Z(h_1) \geq Z(h)$ . This is possible due to Lemma 2.10. Next let  $h_2$  be the symmetrization of  $h_1$  at  $\beta/4$ . Then  $h_2$  is a level 2 snippet of  $h_1$ , and Lemma 2.10 implies that  $Z(h_2) \geq Z(h_1)$ . To get  $h_3$  from  $h_2$ , the symmetrization procedure must be carried out twice, as follows. First let  $g$  be the symmetrization of  $h_2$  at  $\beta/8$ ; then let  $h_3$  be the symmetrization of  $g$  at  $3\beta/8$ . Lemma 2.10 implies that  $Z(h_3) \geq Z(h_2)$ .

This process is carried out inductively. To get  $h_n$  from  $h_{n-1}$ , symmetrization must be carried out  $(2^n - 2^{n-1})/2 = 2^{n-2}$  times, once at

each point of  $(0, \beta/2)$  which is dyadic of level  $n$  but not of level  $n-1$ . This gives a level  $n$  snippet  $h_n$  of  $h_{n-1}$  such that  $Z(h_n) \geq Z(h_{n-1})$ .  $\square$

For  $h : [0, \beta]^p \rightarrow \mathbb{R}$  weakly differentiable, write  $\|h\|' = \|h'\|_\infty$ . The following lemma is immediate from the definition (42).

LEMMA 2.14.  $Z(\cdot)$  is continuous in the norm  $\|\cdot\|'$ .

We can now complete the proof of (43), and hence Lemma 2.3.

*Proof of (43).* Let  $(h_n : n \geq 0)$  be the sequence produced by Lemma 2.13. Note that  $h_n$  is a level  $n$  snippet of  $h$  itself. In particular  $\|h'_n\|_\infty \leq \|h'\|_\infty =: M$  for all  $n \geq 1$ . By symmetry we may assume that  $h_n(t) = h(k_n 2^{-n}\beta + t)$  for all  $0 \leq t < 2^{-n}\beta$  and some  $k_n \in \{0, 1, \dots, 2^n - 1\}$ ; that is,  $a = 0$  in Definition 2.12. Moreover, since  $h_{n+1}$  is a level  $n$  snippet of  $h_n$  we may also assume that  $k_{n+1} \in \{2k_n, 2k_n + 1\}$  for all  $n$ ; that is, the restriction of  $h_{n+1}$  to  $[0, 2^{-(n+1)}\beta)$  equals the restriction of  $h_n$  to either  $[0, 2^{-(n+1)}\beta)$  or to  $[2^{-(n+1)}\beta, 2^{-n}\beta)$ .

Let  $t_n = (k_n + 1/2)2^{-n}\beta$  be the midpoint. Clearly the sequence  $t_n$  is convergent, with limit  $t_*$  say. Let  $r_n = h'_n(2^{-(n+1)}\beta) = h'(t_n)$ . By continuity of  $h'$  we have that  $r_n \rightarrow r := h'(t_*) \in [-M, M]$ . In fact, since  $[0, \beta]^p$  is compact,  $h'$  is uniformly continuous. So for  $\varepsilon > 0$  given we have for large enough  $n$  that  $|h'(s) - h'(t)| < \varepsilon$  whenever  $|s - t| \leq 2^{-n}\beta$ . This implies that  $\|h'_n - W'_{r_n, n}\|_\infty < \varepsilon$  for large enough  $n$ . Since  $\|W'_{r, n} - W'_{r_n, n}\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$  it follows from Lemma 2.14 that for any  $\varepsilon' > 0$  we have

$$|Z(h_n) - Z(W_{r, n})| \leq |Z(h_n) - Z(W_{r_n, n})| + |Z(W_{r_n, n}) - Z(W_{r, n})| < \varepsilon'$$

whenever  $n$  is large enough. Hence by Lemma 2.13,

$$Z(h) \leq Z(h_n) < Z(W_{r, n}) + \varepsilon'$$

for  $n$  large enough. Since  $\varepsilon' > 0$  was arbitrary it follows from Lemma 2.8 that  $Z(h) \leq \zeta(r)Z(0)$ . Since  $\zeta$  is an even function we may assume that  $r \geq 0$ .  $\square$

### 3. MEAN-FIELD BEHAVIOUR OF THE SUSCEPTIBILITY

The main objective of this section is to prove Theorem 1.3, giving the critical exponent value  $\gamma = 1$ . The arguments in this section are inspired by similar arguments for the classical Ising model in [1, 2, 3]. Apart from Theorem 1.2, the main component in the proof is a pair of new differential inequalities for the susceptibility. The proof of these inequalities uses the random-parity representation of [10], which we briefly describe next (see also [11] for the closely related random-current representation).

**3.1. The random-parity representation.** Throughout this subsection and the next,  $\Lambda$  and  $\beta$  will be fixed and finite. Write  $\Lambda \times [0, \beta]^p = K$ . The random-parity representation allows one to write, for each finite set  $A \subseteq K$ ,

$$(58) \quad \mu_\Lambda^\beta \left( \prod_{(x,t) \in A} \sigma(x,t) \right) = \frac{E(\partial\psi^A)}{E(\partial\psi^\emptyset)},$$

where  $\psi^A$  is a certain random labelling of  $K$  using the labels ‘even’ and ‘odd’ with ‘source set’  $A$ , and  $\partial\psi^A$  is a positive weight associated with the labelling. The  $\psi^A$  are constructed using Poisson processes, but the measure  $E$  is *not* the same as in Section 1.1. Throughout Section 3 we will use  $E$  for the random-parity measure only. Elements of  $K$  will simply be denoted by  $x, y, a, b, \dots$ , and we let

$$F = \{xy : x = (u, t) \in K, y = (v, t) \in K \text{ for some } u \sim v \in \Lambda\}$$

be the set of unordered pairs of ‘adjacent’ elements of  $K$ . The point  $(0, 0) \in K$  will be denoted by 0.

The labelling  $\psi^A$  is constructed using a Poisson process  $S$  on  $F$ , of intensity  $\lambda$ . As one traverses each ‘circle’  $\{u\} \times [0, \beta]^p$  in  $K$ , the label alternates between ‘even’ and ‘odd’ in such a way that the label always changes at (i) points  $x \in A$ , and (ii) points  $x$  and  $y$  such that  $xy \in S$ . Moreover, these are the only types of points where the label is allowed to change (this imposes constraints on  $S$ ). See Figure 4 for an illustration of such labellings. One may see that in such a labelling  $\psi^A$

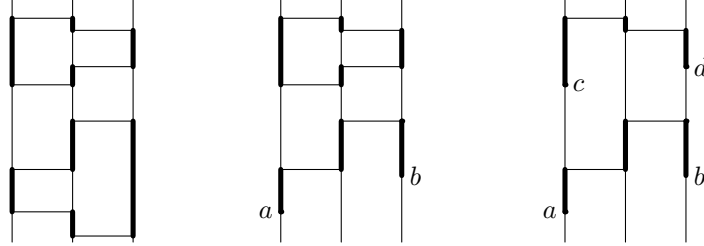


FIGURE 4. Labellings  $\psi^A$  for  $A = \emptyset$  (left),  $A = \{a, b\}$  (middle) and  $A = \{a, b, c, d\}$  (right). Horizontal line segment represent elements of  $S$ . Thick vertical segments are ‘odd’ and thin segments ‘even’.

the ‘odd’ subset of  $K$  forms a collection of geometric ‘paths’ between elements of  $A$ , together with a collection of ‘loops’. A labelling as described above can only be defined if  $A$  has even size; if  $A$  has odd size we define the weight  $\partial\psi^A = 0$  (this is consistent with (58)).

Although there is a natural notion of connectivity along ‘odd’ paths, it turns out to be much more fruitful to consider a more complex notion of connectivity in *triples*  $(\psi_1^A, \psi_2^B, \Delta)$ . Here  $\psi_1^A$  and  $\psi_2^B$  are independent labellings and  $\Delta$  is an independent Poisson process of ‘cuts’ of intensity

4δ. This is described in detail in [10]; here is a brief account. Let  $S_1$  and  $S_2$  denote the (independent) Poisson processes on  $F$  used to construct  $\psi_1^A$  and  $\psi_2^B$ , respectively. Elements  $xy \in S_1 \cup S_2$  may be interpreted as ‘bridges’ which connect the points  $x$  and  $y$ . Connections may traverse such bridges, and may also traverse subintervals of  $K$  *except that* connections are blocked at points  $x \in \Delta$  such that both  $\psi_1^A$  and  $\psi_2^B$  are ‘even’ at  $x$  (‘odd’ labels thus ‘cancel’  $\Delta$ ). For  $a, b \in K$  we write  $\{a \leftrightarrow b\}$  for the event that  $a$  and  $b$  are connected in the triple  $(\psi_1^A, \psi_2^B, \Delta)$ .

There are two main technical tools in the random-parity representation.

- (1) *The switching lemma* [10, Theorem 4.2] implies that for any  $a, b \in K$  and any two finite sets  $A, B \subseteq K$ ,

$$E(\partial\psi_1^A \partial\psi_2^B \mathbb{I}\{a \leftrightarrow b\}) = E(\partial\psi_1^{A\Delta ab} \partial\psi_2^{B\Delta ab} \mathbb{I}\{a \leftrightarrow b\}).$$

Here  $A\Delta ab$  is short-hand for the set-theoretic symmetric difference  $A\Delta\{a, b\}$ . The main manifestation of the switching lemma is the identity

$$E(\partial\psi_1^A \partial\psi_2^{ab}) = E(\partial\psi_1^{A\Delta ab} \partial\psi_2^\emptyset \mathbb{I}\{a \leftrightarrow b\}).$$

(The connection  $a \leftrightarrow b$  is automatic in the left-hand-side since there is an odd path in  $\psi_2^{ab}$ .)

- (2) *Conditioning on clusters*. It is often useful to consider how a triple  $(\psi_1^A, \psi_2^B, \Delta)$  interacts with a third (independent) labelling  $\psi_3^C$ . One may then condition on the set  $C_{1,2}(x)$  of points connected to some given point  $x \in K$  in  $(\psi_1^A, \psi_2^B, \Delta)$ , and consider the restrictions of  $\psi_1^A$ ,  $\psi_2^B$  and  $\psi_3^C$  to  $C_{1,2}(x)$  and  $K \setminus C_{1,2}(x)$  ‘separately’. The details of this procedure are technical and depend on the precise situation in which it is to be used (care must be taken to get the correct ‘sources’ in the restricted labellings). Rather than attempting to describe the details we point to [10, Lemma 4.6 and (5.6)–(5.8)], where applications of this method are described in detail.

For simplicity of notation we will, in this subsection and the next, write  $\langle\sigma_A\rangle$  for the quantity in (58), and will write  $Z = E(\partial\psi^\emptyset)$ . We write  $\langle\sigma_A; \sigma_B\rangle$  for  $\langle\sigma_A\sigma_B\rangle - \langle\sigma_B\rangle\langle\sigma_A\rangle = \langle\sigma_{A\Delta B}\rangle - \langle\sigma_B\rangle\langle\sigma_A\rangle$ .

Here is an example of the random-parity representation in action. Let

$$\chi_\Lambda = \chi_\Lambda(\delta, \lambda, \beta) := \sum_{x \in \Lambda} \int_0^\beta \mu_\Lambda^\beta(\sigma(0, 0)\sigma(x, t)) dt$$

denote the finite-volume, positive-temperature approximation of the susceptibility (12). We see (using the expression in Definition 1.1) that

$$(59) \quad \frac{\partial\chi_\Lambda}{\partial\lambda} = \int_K dx \int_F d(yz) \langle\sigma_0\sigma_x; \sigma_y\sigma_z\rangle.$$

Using the switching lemma,

$$\begin{aligned}
 \langle \sigma_0 \sigma_x; \sigma_y \sigma_z \rangle &= \frac{1}{Z} E(\partial \psi_1^{0xyz}) - \frac{1}{Z^2} E(\partial \psi_1^{0x} \partial \psi_2^{yz}) \\
 (60) \quad &= \frac{1}{Z^2} E(\partial \psi_1^{0xyz} \partial \psi_2^\varnothing) - \frac{1}{Z^2} E(\partial \psi_1^{0xyz} \partial \psi_2^\varnothing \mathbb{I}\{y \leftrightarrow z\}) \\
 &= \frac{1}{Z^2} E(\partial \psi_1^{0xyz} \partial \psi_2^\varnothing \mathbb{I}\{y \not\leftrightarrow z\}).
 \end{aligned}$$

In particular  $\frac{\partial \chi_\Lambda}{\partial \lambda} \geq 0$ .

**3.2. Differential inequalities.** Let

$$B_\Lambda = B_\Lambda(\delta, \lambda, \beta) := \sum_{x \in \Lambda} \int_0^\beta \mu_\Lambda^\beta(\sigma(0, 0) \sigma(x, t))^2 dt$$

be the finite-volume, positive-temperature approximation of the bubble-diagram (13). In addition to Theorem 1.2, the main step in proving Theorem 1.3 is to establish the following two differential inequalities:

LEMMA 3.1. *We have that*

$$(61) \quad 4d\chi_\Lambda^2 \geq \frac{\partial \chi_\Lambda}{\partial \lambda} \geq 4d\chi_\Lambda^2 - 4dB_\Lambda \chi_\Lambda - 2d\lambda B_\Lambda \frac{\partial \chi_\Lambda}{\partial \lambda} - 8d\delta B_\Lambda \left( -\frac{\partial \chi_\Lambda}{\partial \delta} \right)$$

and

$$(62) \quad 2\chi_\Lambda^2 \geq -\frac{\partial \chi_\Lambda}{\partial \delta} \geq 2\chi_\Lambda^2 - 2B_\Lambda \chi_\Lambda - \lambda B_\Lambda \frac{\partial \chi_\Lambda}{\partial \lambda} - 4\delta B_\Lambda \left( -\frac{\partial \chi_\Lambda}{\partial \delta} \right).$$

These inequalities are analogous to inequalities for the classical Ising model in [1], and the proof follows a similar outline. We recall from (60) that  $\frac{\partial \chi_\Lambda}{\partial \lambda} \geq 0$ , and remark that  $\frac{\partial \chi_\Lambda}{\partial \delta} \leq 0$  (see (70) below).

*Proof.* We start with (61). From (59) we have that

$$\begin{aligned}
 (63) \quad \frac{\partial \chi_\Lambda}{\partial \lambda} &= \int_K dx \int_F d(yz) [\langle \sigma_0 \sigma_y \rangle \langle \sigma_x \sigma_z \rangle + \langle \sigma_0 \sigma_z \rangle \langle \sigma_x \sigma_y \rangle + U_4(0, x, y, z)] \\
 &= 4d\chi_\Lambda^2 + \int_K dx \int_F d(yz) U_4(0, x, y, z),
 \end{aligned}$$

where

$$U_4(a, b, c, d) = \langle \sigma_a \sigma_b \sigma_c \sigma_d \rangle - \langle \sigma_a \sigma_b \rangle \langle \sigma_c \sigma_d \rangle - \langle \sigma_a \sigma_c \rangle \langle \sigma_b \sigma_d \rangle - \langle \sigma_a \sigma_d \rangle \langle \sigma_b \sigma_c \rangle$$

is sometimes called the ‘fourth Ursell function’. Note that  $U_4$  is symmetric in its four arguments. We have that  $U_4 \leq 0$ . In fact

$$\begin{aligned}
 &E(\partial \psi_1^{abcd} \partial \psi_2^\varnothing) - E(\partial \psi_1^{ab} \partial \psi_2^{cd}) - E(\partial \psi_1^{ac} \partial \psi_2^{bd}) - E(\partial \psi_1^{ad} \partial \psi_2^{bc}) \\
 &= E(\partial \psi_1^{abcd} \partial \psi_2^\varnothing [1 - \mathbb{I}\{c \leftrightarrow d\} - \mathbb{I}\{b \leftrightarrow d\} - \mathbb{I}\{b \leftrightarrow c\}]),
 \end{aligned}$$

and the quantity in square brackets is either 0 or  $-2$ , the latter occurring if and only if all four points  $a, b, c, d$  are connected. Applying the switching lemma we arrive at the identity

$$U_4(a, b, c, d) = -2 \frac{1}{Z^2} E(\partial\psi_1^{ab} \partial\psi_2^{cd} \mathbb{I}\{a \leftrightarrow c\}).$$

The upper bound in (61) follows.

The lower bound in (61) will be obtained by bounding

$$\frac{1}{Z^2} E(\partial\psi_1^{ab} \partial\psi_2^{cd} \mathbb{I}\{a \leftrightarrow c\}) = \frac{1}{2} |U_4(a, b, c, d)|$$

from above and using (63). Let  $\psi_3^{cd}$  be an independent labelling. In the configuration  $\psi_3^{cd}$  there is an odd path  $\xi_3^{cd}$  from  $c$  to  $d$ , which is called the ‘backbone’ of the configuration (see [10, Section 3.3]). Let  $C_{1,2}(a)$  denote the connected cluster of  $a$  in the triple  $(\psi_1, \psi_2, \Delta)$ . (All connectivities in this proof will refer to this triple.) Conditioning on the cluster  $C_{1,2}(a)$  as in [10, (5.6)–(5.8)] we find that

$$(64) \quad E(\partial\psi_1^{ab} \partial\psi_2^\emptyset \partial\psi_3^{cd} \mathbb{I}\{\xi_3^{cd} \cap C_{1,2}(a) = \emptyset\}) \\ \leq Z E(\partial\psi_1^{ab} \partial\psi_2^\emptyset \langle \sigma_c \sigma_d \rangle_{K \setminus C_{1,2}(a)}),$$

where  $\langle \sigma_c \sigma_d \rangle_{K \setminus C_{1,2}(a)}$  denotes the correlation (58) in the smaller region  $K \setminus C_{1,2}(a)$ . Also as in [10], we further find that

$$E(\partial\psi_1^{ab} \partial\psi_2^\emptyset \langle \sigma_c \sigma_d \rangle_{K \setminus C_{1,2}(a)}) = E(\partial\psi_1^{ab} \partial\psi_2^{cd} \mathbb{I}\{a \not\leftrightarrow c\}).$$

Thus

$$(65) \quad Z E(\partial\psi_1^{ab} \partial\psi_2^{cd} \mathbb{I}\{a \leftrightarrow c\}) \leq E(\partial\psi_1^{ab} \partial\psi_2^\emptyset \partial\psi_3^{cd} \mathbb{I}\{\xi_3^{cd} \cap C_{1,2}(a) \neq \emptyset\}).$$

The rest of the proof of the lower bound in (61) will be based on bounding the right-hand side of (65). There are two main cases to consider in (65), namely whether or not  $c \in C_{1,2}(a)$ . In case  $c \in C_{1,2}(a)$  we get

$$E(\partial\psi_1^{ab} \partial\psi_2^\emptyset \partial\psi_3^{cd} \mathbb{I}\{c \in C_{1,2}(a)\}) = Z \langle \sigma_c \sigma_d \rangle E(\partial\psi_1^{ab} \partial\psi_2^\emptyset \mathbb{I}\{a \leftrightarrow c\}) \\ = Z^3 \langle \sigma_c \sigma_d \rangle \langle \sigma_b \sigma_c \rangle \langle \sigma_a \sigma_c \rangle.$$

The case  $c \notin C_{1,2}(a)$  splits into two subcases, because the first point  $u$  on  $\xi_3^{cd}$  in  $C_{1,2}(a)$  is then either (i) an endpoint of a bridge on  $\xi_3^{cd}$  whose other endpoint  $v$  is not in  $C_{1,2}(a)$ , or (ii) a point of  $\Delta$  on the boundary of  $C_{1,2}(a)$ . See Figure 5.

Let us first consider the case when  $u$  is the endpoint of a bridge. Then  $\xi_3^{cd}$  decomposes as  $\zeta \circ \zeta'$ , where  $\zeta : c \rightarrow v$  and  $\zeta' : u \rightarrow d$ . Moreover,  $\zeta \cap C_{1,2}(a) = \emptyset$ . We may therefore argue as in (64)–(65) for  $\zeta$ . Using also [10, Lemma 3.3] and the GKS-inequality (proved for the present model in [9, Lemma 2.2.20]), we may therefore in this case



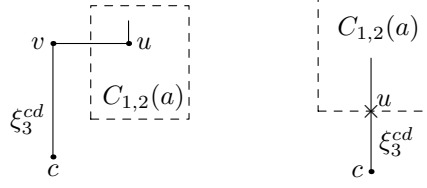


FIGURE 5. Cases for the first point  $u$  on  $\xi_3^{cd}$  in  $C_{1,2}(a)$ . Left: (i)  $u$  is the endpoint of a bridge. Right: (ii)  $u$  is at a ‘cut’ (point of  $\Delta$ ).

bound the right-hand-side of (65) above by

$$\begin{aligned}
 (66) \quad & \lambda \int_F d(uv) \langle \sigma_u \sigma_d \rangle E(\partial \psi_1^{ab} \partial \psi_2^{\varnothing} \langle \sigma_c \sigma_v \rangle_{K \setminus C_{1,2}(a)} \mathbb{I}\{u \in C_{1,2}(a)\}) \\
 & = \lambda Z \int_F d(uv) \langle \sigma_u \sigma_d \rangle E(\partial \psi_1^{ab} \partial \psi_2^{cv} \mathbb{I}\{c \notin C_{1,2}(a), u \in C_{1,2}(a)\}).
 \end{aligned}$$

By the switching lemma, the latter expectation equals

$$E(\partial \psi_1^{bu} \partial \psi_2^{aucv} \mathbb{I}\{c \notin C_{1,2}(a), u \in C_{1,2}(a)\}).$$

By conditioning on the cluster  $C_{1,2}(a)$  and using the GKS-inequality, this is at most  $\langle \sigma_a \sigma_u \rangle E(\partial \psi_1^{bu} \partial \psi_2^{cv} \mathbb{I}\{u \not\leftrightarrow v\})$ .

Let us now consider the case when the first point  $u$  on  $\xi_3^{cd}$  in  $C_{1,2}(a)$  is a point of  $\Delta$  on the boundary of  $C_{1,2}(a)$ . Then  $\xi_3^{cd}$  decomposes as  $\zeta \circ \zeta'$ , where  $\zeta : c \rightarrow u$ ,  $\zeta' : u \rightarrow d$  and  $\zeta \cap C_{1,2}(a) = \{u\}$ . If  $u$  is removed from  $\Delta$  then  $C_{1,2}(a)$  is enlarged, and what was previously  $C_{1,2}(a)$  becomes  $C_{1,2}^u(a)$ , the set of points which can be reached from  $a$  without passing  $u$ . We may argue as for (65) and (66) again to see that the right-hand-side of (65) is in this case at most

$$\begin{aligned}
 & 4\delta \int_K du \langle \sigma_u \sigma_d \rangle E(\partial \psi_1^{ab} \partial \psi_2^{\varnothing} \langle \sigma_c \sigma_u \rangle_{K \setminus C_{1,2}^u(a)} \mathbb{I}\{b \in C_{1,2}^u(a), u \in C_{1,2}(a)\}) \\
 & = 4\delta Z \int_K du \langle \sigma_u \sigma_d \rangle E(\partial \psi_1^{ab} \partial \psi_2^{cu} \mathbb{I}\{c \notin C_{1,2}^u(a), b \in C_{1,2}^u(a), u \in C_{1,2}(a)\}).
 \end{aligned}$$

By the switching lemma, the latter expectation is at most

$$\begin{aligned}
 E(\partial \psi_1^{bu} \partial \psi_2^{ac} \mathbb{I}\{c \notin C_{1,2}^u(a), b \in C_{1,2}^u(a)\}) & = E(\partial \psi_1^{bc} \partial \psi_2^{au} \mathbb{I}\{c \notin C_{1,2}^u(a), b \in C_{1,2}^u(a)\}) \\
 & \leq \langle \sigma_a \sigma_u \rangle E(\partial \psi_1^{bc} \partial \psi_2^{\varnothing} \mathbb{I}\{b \overset{u}{\leftrightarrow} c\}),
 \end{aligned}$$

where we have conditioned on the cluster  $C_{1,2}^u(a)$  and used the GKS-inequality for the upper bound.

So far we have established that

$$(67) \quad (0 \leq) -\frac{1}{2}U_4(a, b, c, d) \leq \langle \sigma_a \sigma_c \rangle \langle \sigma_b \sigma_c \rangle \langle \sigma_c \sigma_d \rangle \\ + \lambda \int_F d(uv) \langle \sigma_a \sigma_u \rangle \langle \sigma_d \sigma_u \rangle E(\partial \psi_1^{bu} \partial \psi_2^{cv} \mathbb{I}\{u \not\leftrightarrow v\}) \\ + 4\delta \int_K du \langle \sigma_a \sigma_u \rangle \langle \sigma_d \sigma_u \rangle E(\partial \psi_1^{bc} \partial \psi_2^{\varnothing} \mathbb{I}\{b \overset{u}{\leftrightarrow} c\}).$$

Whereas  $U_4(a, b, c, d)$  is symmetric in  $a, b, c, d$  the right-hand-side of (67) is not. Averaging with respect to the transposition  $b \leftrightarrow c$  we arrive at the upper bound

$$(68) \quad -U_4(a, b, c, d) \leq \langle \sigma_a \sigma_c \rangle \langle \sigma_b \sigma_c \rangle \langle \sigma_c \sigma_d \rangle + \langle \sigma_a \sigma_b \rangle \langle \sigma_b \sigma_c \rangle \langle \sigma_b \sigma_d \rangle \\ + \lambda \int_F d(uv) \langle \sigma_a \sigma_u \rangle \langle \sigma_d \sigma_u \rangle [E(\partial \psi_1^{bu} \partial \psi_2^{cv} \mathbb{I}\{u \not\leftrightarrow v\}) + E(\partial \psi_1^{cu} \partial \psi_2^{bv} \mathbb{I}\{u \not\leftrightarrow v\})] \\ + 8\delta \int_K du \langle \sigma_a \sigma_u \rangle \langle \sigma_d \sigma_u \rangle E(\partial \psi_1^{bc} \partial \psi_2^{\varnothing} \mathbb{I}\{b \overset{u}{\leftrightarrow} c\}).$$

Thus, setting  $a = y, b = x, c = 0, d = z$ , it follows that the quantity

$$-\int_K dx \int_F d(yz) U_4(0, x, y, z)$$

which appears in (63) is at most

$$(69) \quad \chi_\Lambda \int_F d(yz) \langle \sigma_0 \sigma_y \rangle \langle \sigma_0 \sigma_z \rangle + \int_K dx \langle \sigma_0 \sigma_x \rangle \int_F d(yz) \langle \sigma_x \sigma_y \rangle \langle \sigma_x \sigma_z \rangle \\ + \lambda \int_K dx \int_F d(uv) [E(\partial \psi_1^{xu} \partial \psi_2^{0v} \mathbb{I}\{u \not\leftrightarrow v\}) + E(\partial \psi_1^{0u} \partial \psi_2^{xv} \mathbb{I}\{u \not\leftrightarrow v\})] \int_F d(yz) \langle \sigma_y \sigma_u \rangle \langle \sigma_z \sigma_u \rangle \\ + 8\delta \int_K dx \int_K du E(\partial \psi_1^{0x} \partial \psi_2^{\varnothing} \mathbb{I}\{0 \overset{u}{\leftrightarrow} x\}) \int_F d(yz) \langle \sigma_y \sigma_u \rangle \langle \sigma_z \sigma_u \rangle.$$

The Cauchy–Schwarz inequality implies that

$$\int_F d(yz) \langle \sigma_0 \sigma_y \rangle \langle \sigma_0 \sigma_z \rangle \leq 2dB_\Lambda;$$

using this together with translation invariance and (60) shows that the quantity in (69) is at most

$$4d\chi_\Lambda B_\Lambda + 2d\lambda B_\Lambda \frac{\partial \chi_\Lambda}{\partial \lambda} + 8d\delta B_\Lambda \left( -\frac{\partial \chi_\Lambda}{\partial \delta} \right).$$

Together with (63), this proves the lower bound in (61).

For the upper bound in (62) we note that

$$(70) \quad -\frac{\partial \chi_\Lambda}{\partial \delta} = 2 \int_K dx \int_K dy \frac{1}{Z^2} E(\partial \psi_1^{0x} \partial \psi_2^{\varnothing} \mathbb{I}\{0 \overset{y}{\leftrightarrow} x\})$$

where the notation  $0 \xleftrightarrow{y} x$  signifies that *if* a path connects 0 and  $x$ , then it must contain  $y$  (see [10, Theorem 4.10]). By the switching lemma

$$\frac{1}{Z^2} E(\partial\psi_1^{0x} \partial\psi_2^\varnothing \mathbb{I}\{0 \xleftrightarrow{y} x\}) = \frac{1}{Z^2} E(\partial\psi_1^{0y} \partial\psi_2^{yx} \mathbb{I}\{0 \xleftrightarrow{y} x\}) \leq \langle \sigma_0 \sigma_y \rangle \langle \sigma_y \sigma_x \rangle.$$

Together with (70) and translation invariance, this proves the upper bound in (62).

The lower bound in (62) is similar in spirit to the lower bound in (61), but differs in the details. We start by recalling that

$$\begin{aligned} (71) \quad -\frac{\partial \chi_\Lambda}{\partial \delta} &= 2 \int_K dx \int_K dy \frac{1}{Z^2} E(\partial\psi_1^{0x} \partial\psi_2^\varnothing \mathbb{I}\{0 \xleftrightarrow{y} x\}) \\ &= 2\chi_\Lambda^2 - \frac{2}{Z^2} \int_K dx \int_K dy E(\partial\psi_1^{0x} \partial\psi_2^\varnothing \mathbb{I}\{0 \xleftrightarrow{y} x\}^c). \end{aligned}$$

In words, the event  $\{0 \xleftrightarrow{y} x\}^c$  is that there is some path from 0 to  $x$  which avoids  $y$ . By the switching lemma and the method for (65) we have that

$$\begin{aligned} &\frac{1}{Z^3} E(\partial\psi_1^{0y} \partial\psi_2^\varnothing \partial\psi_3^{xy} \mathbb{I}\{\xi_3^{xy} \cap C_{1,2}^y(0) = \{y\}\}) \\ &\leq \frac{1}{Z^2} E(\partial\psi_1^{0y} \partial\psi_2^\varnothing \langle \sigma_x \sigma_y \rangle_{K \setminus C_{1,2}^y(0)}) \\ &= \frac{1}{Z^2} E(\partial\psi_1^{0y} \partial\psi_2^{xy} \mathbb{I}\{0 \xleftrightarrow{y} x\}) = \frac{1}{Z^2} E(\partial\psi_1^{0x} \partial\psi_2^\varnothing \mathbb{I}\{0 \xleftrightarrow{y} x\}). \end{aligned}$$

Note that certainly  $0 \leftrightarrow y$  in  $(\psi_1^{0y}, \psi_2^\varnothing, \Delta)$ , since  $\psi_1$  has sources 0,  $y$ ; so in this situation the complement of the event  $\{\xi_3^{xy} \cap C_{1,2}^y(0) = \{y\}\}$  is the event  $\{\xi_3^{xy} \cap C_{1,2}^y(0) \supsetneq \{y\}\}$  that  $\{y\}$  is a strict subset of  $\xi_3^{xy} \cap C_{1,2}^y(0)$ . Thus

$$(72) \quad \frac{1}{Z^2} E(\partial\psi_1^{0x} \partial\psi_2^\varnothing \mathbb{I}\{0 \xleftrightarrow{y} x\}^c) \leq \frac{1}{Z^3} E(\partial\psi_1^{0y} \partial\psi_2^\varnothing \partial\psi_3^{xy} \mathbb{I}\{\xi_3^{xy} \cap C_{1,2}^y(0) \supsetneq \{y\}\}).$$

We consider the cases whether or not  $x \in C_{1,2}(0)$  in the expectation in the right-hand-side of (72); note that  $C_{1,2}^y(0) \subseteq C_{1,2}(0)$ . The case  $x \in C_{1,2}(0)$  gives at most

$$E(\partial\psi_1^{0y} \partial\psi_2^\varnothing \partial\psi_3^{xy} \mathbb{I}\{x \in C_{1,2}(0)\}) = Z^3 \langle \sigma_x \sigma_y \rangle^2 \langle \sigma_0 \sigma_x \rangle.$$

Again, the case  $x \notin C_{1,2}(0)$  decomposes into the subcases when the first point  $u$  on  $\xi_3^{xy}$  which lies in  $C_{1,2}(0)$  is (i) the endpoint of a bridge whose other endpoint  $v$  does not lie in  $C_{1,2}(0)$ , or (ii) a cut in  $\Delta$  on the boundary of  $C_{1,2}(0)$ .

In case (i), the backbone  $\xi_3^{xy}$  decomposes as  $\zeta \circ \zeta'$  where  $\zeta : x \rightarrow v$ ,  $\zeta' : u \rightarrow y$  and  $\zeta \cap C_{1,2}(0) = \emptyset$ . As for (66) it follows that the

expectation in the right-hand-side of (72) is in this case at most

$$\begin{aligned}
(73) \quad & \lambda Z \int_F d(uv) \langle \sigma_u \sigma_y \rangle E(\partial \psi_1^{0y} \partial \psi_2^{xv} \mathbb{I}\{u \leftrightarrow 0, v \not\leftrightarrow 0\}) \\
&= \lambda Z \int_F d(uv) \langle \sigma_u \sigma_y \rangle E(\partial \psi_1^{uy} \partial \psi_2^{0uv} \mathbb{I}\{u \not\leftrightarrow v, u \not\leftrightarrow x\}) \\
&\leq \lambda Z \int_F d(uv) \langle \sigma_u \sigma_y \rangle^2 E(\partial \psi_1^{xv} \partial \psi_2^{0u} \mathbb{I}\{u \not\leftrightarrow v\}).
\end{aligned}$$

In case (ii) the backbone  $\xi_3^{xy}$  decomposes as  $\zeta \circ \zeta'$ , where  $\zeta : x \rightarrow u$ ,  $\zeta' : u \rightarrow y$  and  $\zeta \cap C_{1,2}(0) = \{u\}$ . The expectation in the right-hand-side of (72) is in this case at most

$$\begin{aligned}
(74) \quad & 4\delta Z \int_K du \langle \sigma_u \sigma_y \rangle E(\partial \psi_1^{0y} \partial \psi_2^{xu} \mathbb{I}\{0 \leftrightarrow u, 0 \overset{u}{\leftrightarrow} x\}) \\
&\leq 4\delta Z \int_K du \langle \sigma_u \sigma_y \rangle^2 E(\partial \psi_1^{0x} \partial \psi_2^{\varnothing} \mathbb{I}\{0 \overset{u}{\leftrightarrow} x\}).
\end{aligned}$$

So far we have showed that

$$\begin{aligned}
(75) \quad & \frac{1}{Z^2} E(\partial \psi_1^{0x} \partial \psi_2^{\varnothing} \mathbb{I}\{0 \overset{y}{\leftrightarrow} x\}^c) \\
&\leq \langle \sigma_x \sigma_y \rangle^2 \langle \sigma_0 \sigma_x \rangle + \lambda \int_F d(uv) \langle \sigma_u \sigma_y \rangle^2 \frac{1}{Z^2} E(\partial \psi_1^{xv} \partial \psi_2^{0u} \mathbb{I}\{u \not\leftrightarrow v\}) \\
&\quad + 4\delta \int_K du \langle \sigma_u \sigma_y \rangle^2 \frac{1}{Z^2} E(\partial \psi_1^{0x} \partial \psi_2^{\varnothing} \mathbb{I}\{0 \overset{u}{\leftrightarrow} x\}).
\end{aligned}$$

Whereas the left-hand-side in (75) is symmetric under the transposition  $0 \leftrightarrow x$ , the right-hand-side is not. Averaging with respect to this transposition we see that we may replace the right-hand-side in (75) by

$$\begin{aligned}
(76) \quad & \frac{1}{2} (\langle \sigma_x \sigma_y \rangle^2 \langle \sigma_0 \sigma_x \rangle + \langle \sigma_0 \sigma_y \rangle^2 \langle \sigma_0 \sigma_x \rangle) \\
&+ \frac{\lambda}{2} \int_F d(uv) \langle \sigma_u \sigma_y \rangle^2 \frac{1}{Z^2} [E(\partial \psi_1^{xv} \partial \psi_2^{0u} \mathbb{I}\{u \not\leftrightarrow v\}) + E(\partial \psi_1^{0v} \partial \psi_2^{xu} \mathbb{I}\{u \not\leftrightarrow v\})] \\
&\quad + 4\delta \int_K du \langle \sigma_u \sigma_y \rangle^2 \frac{1}{Z^2} E(\partial \psi_1^{0x} \partial \psi_2^{\varnothing} \mathbb{I}\{0 \overset{u}{\leftrightarrow} x\}).
\end{aligned}$$

It follows that the integral

$$2 \int_K dx \int_K dy \frac{1}{Z^2} E(\partial \psi_1^{0x} \partial \psi_2^{\varnothing} \mathbb{I}\{0 \overset{y}{\leftrightarrow} x\}^c)$$

which appears in (71) is at most

$$2B_\Lambda \chi_\Lambda + \lambda B_\Lambda \frac{\partial \chi_\Lambda}{\partial \lambda} + 4\delta B_\Lambda \left( -\frac{\partial \chi_\Lambda}{\partial \delta} \right).$$

This proves the lower bound in (62).  $\square$

**3.3. The bubble-diagram.** This section is devoted to bounds on the bubble-diagram (13). For each  $\delta \geq 0$  and  $0 < \beta \leq \infty$  one may define the critical value  $\lambda_c = \lambda_c(\delta, \beta)$  by

$$\begin{aligned}\lambda_c(\delta, \beta) &= \sup \left\{ \lambda > 0 : \limsup_{N \rightarrow \infty} \chi_\Lambda(\lambda, \delta, \beta) < \infty \right\}, \quad \text{if } \beta < \infty, \\ \lambda_c(\delta, \infty) &= \sup \left\{ \lambda > 0 : \limsup_{N, \beta \rightarrow \infty} \chi_\Lambda(\lambda, \delta, \beta) < \infty \right\}.\end{aligned}$$

This is the the same critical value as referred to in Section 1, see [10, Theorem 1.1]. In all simultaneous limits  $N, \beta \rightarrow \infty$  we assume that the limit is taken so that  $\beta$  and  $N$  are of the same order (this is convenient when using the graphical representation and is related to ‘van Hove convergence’). It is well-known that  $0 < \lambda_c < \infty$  provided either  $\beta < \infty$  and  $d \geq 2$ , or  $\beta = \infty$  and  $d \geq 1$ .

For each  $0 < \beta \leq \infty$  and  $\lambda < \lambda_c$  there is a unique infinite-volume limit measure of  $\mu_\Lambda^\beta$ ; for  $\beta = \infty$  it is obtained in the simultaneous limit  $N, \beta \rightarrow \infty$ . For simplicity we will denote this limit measure by  $\mu$ . From the dominated convergence theorem it follows that, as  $N \rightarrow \infty$  or  $N, \beta \rightarrow \infty$ , the limits  $\chi$  and  $B$  of  $\chi_\Lambda$  and  $B_\Lambda$ , respectively, exist, and that

$$\chi = \begin{cases} \sum_{x \in \mathbb{Z}^d} \int_0^\beta \mu(\sigma(0, 0) \sigma(x, t)) dt, & \text{if } \beta < \infty, \\ \sum_{x \in \mathbb{Z}^d} \int_{-\infty}^\infty \mu(\sigma(0, 0) \sigma(x, t)) dt, & \text{if } \beta = \infty, \end{cases}$$

and

$$B = \begin{cases} \sum_{x \in \mathbb{Z}^d} \int_0^\beta \mu(\sigma(0, 0) \sigma(x, t))^2 dt, & \text{if } \beta < \infty, \\ \sum_{x \in \mathbb{Z}^d} \int_{-\infty}^\infty \mu(\sigma(0, 0) \sigma(x, t))^2 dt, & \text{if } \beta = \infty. \end{cases}$$

By [10, Theorem 6.3],  $\chi \uparrow \infty$  as  $\lambda \uparrow \lambda_c$ . Since  $\mu(\sigma(0, 0) \sigma(x, t)) \leq 1$  we have  $B \leq \chi$  and hence  $B$  may or may not diverge at  $\lambda_c$ . In the following statement,  $\log_+ \chi$  is shorthand for  $(\log \chi) \vee 0$ .

**LEMMA 3.2.** *Let  $0 < \beta \leq \infty$  and assume that  $\delta \geq \varepsilon_0$  and  $\varepsilon_0 \leq \lambda < \lambda_c$  for some fixed  $\varepsilon_0 > 0$ .*

- (1) *Suppose either (a)  $\beta < \infty$  and  $d > 4$ , or (b)  $\beta = \infty$  and  $d > 3$ . Then there is a finite constant  $C_1 = C_1(\varepsilon_0, \beta)$  such that  $B(\lambda, \delta, \beta) \leq C_1$ .*
- (2) *Suppose either (a)  $\beta < \infty$  and  $d = 4$ , or (b)  $\beta = \infty$  and  $d = 3$ . Then there is a finite constant  $C_2 = C_2(\varepsilon_0, \beta)$  such that  $B(\lambda, \delta, \beta) \leq C_2(1 + \log_+ \chi)$ .*

*Proof.* We start by proving the first statement. Write  $\hat{K} = (\frac{2\pi}{2N}\Lambda) \times (\frac{2\pi}{\beta}\mathbb{Z})$  and  $\hat{K}^\times = \hat{K} \setminus (0, 0)$ . By Plancherel’s formula and the infrared

bound,

$$\begin{aligned} B_\Lambda &= \frac{1}{\beta|\Lambda|} \sum_{(k,l) \in \hat{K}} \hat{c}_\Lambda(k,l)^2 = \frac{1}{\beta|\Lambda|} \left[ \chi_\Lambda^2 + \sum_{(k,l) \in \hat{K}^\times} \hat{c}_\Lambda(k,l)^2 \right] \\ &\leq \frac{1}{\beta|\Lambda|} \left[ \chi_\Lambda^2 + \sum_{(k,l) \in \hat{K}^\times} \left( \frac{48}{2\lambda\hat{L}(k) + l^2/2\delta} \right)^2 \right]. \end{aligned}$$

We conclude that for all  $\lambda < \lambda_c$ , the bubble diagram  $B$  is at most

$$(77) \quad \frac{1}{(2\pi)^d \beta} \sum_{l \in \frac{2\pi}{\beta} \mathbb{Z}} \int_{(-\pi, \pi]^d} \left( \frac{48}{2\lambda\hat{L}(k) + l^2/2\delta} \right)^2 dk, \quad \text{if } \beta < \infty,$$

and at most

$$(78) \quad \frac{1}{(2\pi)^{d+1}} \int_{-\infty}^{\infty} \int_{(-\pi, \pi]^d} \left( \frac{48}{2\lambda\hat{L}(k) + l^2/2\delta} \right)^2 dl dk, \quad \text{if } \beta = \infty.$$

For any  $a > 0$ ,

$$(79) \quad \sum_{l \in \mathbb{Z}} \frac{1}{(a + l^2)^2} \leq \frac{1}{a^2} + \int_{-\infty}^{\infty} \frac{1}{(a + l^2)^2} dl = \frac{1}{a^2} + \frac{\pi/2}{a^{3/2}}.$$

Applying this with  $a(k) = 4\lambda\delta\hat{L}(k)$  we deduce that the quantity in (77) is at most an absolute constant times

$$(80) \quad \frac{\delta^2}{\beta} \int_{(-\pi, \pi]^d} \left( \frac{1}{a(k)^2} + \frac{1}{a(k)^{3/2}} \right) dk$$

and that the quantity in (78) is at most an absolute constant times

$$\delta^2 \int_{(-\pi, \pi]^d} \frac{1}{a(k)^{3/2}} dk.$$

Recall that  $\hat{L}(k) = \sum_{j=1}^d (1 - \cos(k_j))$ . In particular,  $\hat{L}(k) \rightarrow 0$  as  $k \rightarrow 0$  but  $\hat{L}(k)$  is positive for all nonzero  $k \in (-\pi, \pi]^d$ . The dominant term in (80) as  $k \rightarrow 0$  is  $1/a(k)^2$ . Hence there is a constant  $c = c(\varepsilon_0, \beta)$  such that

$$B \leq c \int_{(-\pi, \pi]^d} \frac{dk}{\hat{L}(k)^\alpha},$$

where  $\alpha = 2$  if  $\beta < \infty$  and  $\alpha = 3/2$  if  $\beta = \infty$ . There is also a constant  $c' = c'(d)$  such that  $\hat{L}(k) \geq c'\|k\|_2^2$  for all  $k \in (-\pi, \pi]^d$ . By using polar coordinates we see that

$$\int_{(-\pi, \pi]^d} \frac{dk}{\|k\|_2^{2\alpha}} \leq \int_0^{2\pi} \frac{r^{d-1}}{r^{2\alpha}} dr,$$

which is finite if  $d > 2\alpha$ . Thus for  $d > 2\alpha$  and for all  $\lambda < \lambda_c$ , we have that  $B \leq C_1(\varepsilon_0, \beta) = c(\varepsilon_0, \beta)/c'(d) \int_0^{2\pi} r^{d-1-2\alpha} dr < \infty$ . This proves the first statement.

We now prove the second statement. By the triangle inequality and the nonnegativity of  $\hat{c}(k, l)$  we have that  $|\hat{c}_\Lambda(k, l)| \leq \chi_\Lambda$  for all  $(k, l) \in \hat{K}$ . Thus

$$|\hat{c}_\Lambda(k, l)| \leq \min \left( \chi_\Lambda, \frac{48}{2\lambda\hat{L}(k) + l^2/2\delta} \right) \leq \frac{2}{\chi_\Lambda^{-1} + (2\lambda\hat{L}(k) + l^2/2\delta)/48}.$$

Hence there is a constant  $c = c(\varepsilon_0, \beta)$  such that

$$(81) \quad B \leq c \sum_{l \in \frac{2\pi}{\beta}\mathbb{Z}} \int_{(-\pi, \pi]^d} \left( \frac{1}{\chi^{-1} + \hat{L}(k) + l^2} \right)^2 dk, \quad \text{if } d = 4 \text{ and } \beta < \infty,$$

and

$$(82) \quad B \leq c \int_{-\infty}^{\infty} \int_{(-\pi, \pi]^d} \left( \frac{1}{\chi^{-1} + \hat{L}(k) + l^2} \right)^2 dl dk, \quad \text{if } d = 3 \text{ and } \beta = \infty.$$

As in the first part it follows that there is a constant  $c'(\varepsilon_0, \beta)$  such that

$$B \leq c' \int_0^{2\pi} \frac{r^3}{(\chi^{-1} + r^2)^2} dr, \quad \text{if } d = 4 \text{ and } \beta < \infty,$$

and

$$B \leq c' \int_0^{2\pi} \frac{r^2}{(\chi^{-1} + r^2)^{3/2}} dr, \quad \text{if } d = 3 \text{ and } \beta = \infty.$$

By explicit computation of these integrals (or otherwise) the result follows.  $\square$

**3.4. Critical exponents.** We now turn to the proof of Theorem 1.3. We split the result into two propositions, one for the upper bound and one for the lower bound, with some additional details added. Note that  $\chi$  is (weakly) increasing in  $\lambda$  and decreasing in  $\delta$  (this can be seen, for example, in (59), (60) and (70)). In particular,  $\lambda_c(\delta)$  is weakly increasing in  $\delta$ . Recall also that  $\chi(\lambda, \delta) \uparrow \infty$  as  $\lambda \uparrow \lambda_c(\delta)$ .

The lower bound does not depend on Lemma 3.2 and is valid whenever  $0 < \lambda_c < \infty$ , which we recall is the case whenever either  $d \geq 2$ , or  $d = 1$  and  $\beta = \infty$ . Here and in what follows  $\|(a, b) - (c, d)\|$  denotes the Euclidean distance between points  $(a, b), (c, d) \in \mathbb{R}^2$ .

**PROPOSITION 3.3.** *Suppose that  $0 < \lambda_c < \infty$ . Then*

- (1)  $\chi$  is continuous in  $(\lambda, \delta)$  whenever  $\lambda < \lambda_c(\delta)$ ;
- (2) For all  $\delta_0 \geq 0$  and all  $\delta \geq \delta_0$  and  $0 \leq \lambda \leq \lambda_c(\delta_0)$  we have that

$$\chi(\delta, \lambda) \geq \frac{1/(\sqrt{3}(4d+2))}{\|(\delta, \lambda) - (\delta_0, \lambda_c(\delta_0))\|}.$$

*Proof.* We use the upper bounds in Lemma 3.1. These may be rewritten as

$$-\frac{\partial \chi_\Lambda^{-1}}{\partial \lambda} \leq 4d \quad \text{and} \quad \frac{\partial \chi_\Lambda^{-1}}{\partial \delta} \leq 2$$

Let  $0 \leq \delta_1 < \delta_2$  and  $0 \leq \lambda_1 < \lambda_2$ . Then

$$\begin{aligned} \chi_\Lambda^{-1}(\delta_2, \lambda_1) - \chi_\Lambda^{-1}(\delta_1, \lambda_2) &= \int_{\delta_1}^{\delta_2} \frac{\partial \chi_\Lambda^{-1}}{\partial \delta} d\delta - \int_{\lambda_1}^{\lambda_2} \frac{\partial \chi_\Lambda^{-1}}{\partial \lambda} d\lambda \\ &\leq (4d+2)(\delta_2 - \delta_1 + \lambda_2 - \lambda_1) \\ &\leq \sqrt{3}(4d+2)\|(\delta_2, \lambda_1) - (\delta_1, \lambda_2)\|. \end{aligned}$$

Now let  $N \rightarrow \infty$  or  $N, \beta \rightarrow \infty$  as appropriate. The continuity statement follows immediately, and the second statement follows on letting  $\delta_1 = \delta_0$  and  $\lambda_2 \uparrow \lambda_c(\delta_0)$ .  $\square$

We now turn to the upper bound in Theorem 1.3. We will in what follows assume that  $\delta, \lambda > \varepsilon_0$  for some arbitrary but fixed  $\varepsilon_0 > 0$ . By continuity we have the following.

**LEMMA 3.4.** *For each  $\delta_0 > \varepsilon_0$  and each  $C > 0$  there is an open, bounded neighbourhood  $U$  of  $(\delta_0, \lambda_c(\delta_0))$  such that  $\chi(\delta, \lambda) \geq 2C$  for all  $(\delta, \lambda) \in U$ .*

**PROPOSITION 3.5.** *Fix  $\theta \in (0, \infty)$  and  $\delta_0 > 0$ .*

- (1) *Suppose that either  $\beta < \infty$  and  $d > 4$ , or  $\beta = \infty$  and  $d > 3$ . There is a neighbourhood  $U$  of  $(\delta_0, \lambda_c(\delta_0))$  and a constant  $c(\delta_0, \theta)$  such that for all  $(\delta, \lambda) \in U$ , of the form  $\delta = \delta' - t$ ,  $\lambda = \lambda_c(\delta') + \theta t$  with  $t < 0$ , we have that  $\chi(\lambda, \delta) \leq c(\delta_0, \theta)/|t|$ .*
- (2) *Suppose that either  $\beta < \infty$  and  $d = 4$ , or  $\beta = \infty$  and  $d = 3$ . There is a neighbourhood  $U$  of  $(\delta_0, \lambda_c(\delta_0))$  and a constant  $c(\delta_0, \theta)$  such that for all  $(\delta, \lambda) \in U$ , of the form  $\delta = \delta' - t$ ,  $\lambda = \lambda_c(\delta') + \theta t$  with  $t < 0$ , we have that  $\chi(\lambda, \delta) \leq c(\delta_0, \theta)|\log t|/|t|$ .*

*Proof.* Throughout the proof we will let  $(\delta, \lambda)$  be of the form  $\delta = \delta' - t$ ,  $\lambda = \lambda_c(\delta') + \theta t$  with  $t < 0$ , and will write  $\chi_\Lambda(t)$  and  $B_\Lambda(t)$  for  $\chi_\Lambda(\delta, \lambda)$  and  $B_\Lambda(\delta, \lambda)$ , respectively. We have that the derivative in  $t$

$$\chi'_\Lambda(t) = \theta \frac{\partial \chi_\Lambda}{\partial \lambda} + \left( - \frac{\partial \chi_\Lambda}{\partial \delta} \right).$$

From (61) and (62) we see that

$$(83) \quad \chi'_\Lambda(t) \geq (4d\theta + 2)\chi_\Lambda(t)^2 \frac{1 - B_\Lambda(t)/\chi_\Lambda(t)}{1 + (2d\lambda + 4\delta + \lambda/\theta + 8d\delta\theta)B_\Lambda(t)}.$$

Restricting  $(\delta, \lambda)$  to an arbitrary bounded open set  $U'$  containing  $(\delta_0, \lambda_c(\delta_0))$  we may replace  $2d\lambda + 4\delta + \lambda/\theta + 8d\delta\theta$  by a uniform upper bound  $c_1(\delta_0, \theta)$ . Since

$$\chi'_\Lambda(t)/\chi_\Lambda(t)^2 = -\frac{d}{dt} \frac{1}{\chi_\Lambda(t)}$$



it follows on integrating that for all  $t_1 < t_2 < 0$  (such that the corresponding points  $(\delta, \lambda)$  lie in  $U'$ ) we have

$$\frac{1}{\chi_\Lambda(t_1)} - \frac{1}{\chi_\Lambda(t_2)} \geq (4d\theta + 2) \int_{t_1}^{t_2} \frac{1 - B_\Lambda(t)/\chi_\Lambda(t)}{1 + c_1(\delta_0, \theta)B_\Lambda(t)} dt.$$

Letting  $\Lambda \uparrow \mathbb{Z}^d$  and applying Fatou's Lemma, we see that

$$(84) \quad \frac{1}{\chi(t_1)} - \frac{1}{\chi(t_2)} \geq (4d\theta + 2) \int_{t_1}^{t_2} \frac{1 - B(t)/\chi(t)}{1 + c_1(\delta_0, \theta)B(t)} dt.$$

For the first part of the statement, let  $U$  be as in Lemma 3.4 with  $C = C_1(\varepsilon_0, \beta)$  of Lemma 3.2. From (84) we see that

$$\frac{1}{\chi(t_1)} - \frac{1}{\chi(t_2)} \geq (t_2 - t_1) \frac{2d\theta + 1}{1 + c_1(\delta_0, \theta)C_1(\varepsilon_0, \beta)}.$$

Letting  $t_2 \uparrow 0$  gives the result.

For the second part, we deduce from (84) and the second part of Lemma 3.2 that

$$\frac{1}{\chi(t_1)} - \frac{1}{\chi(t_2)} \geq (4d\theta + 2) \int_{t_1}^{t_2} \frac{1 - C_2(\varepsilon_0, \beta)(1 + \log \chi(t))/\chi(t)}{1 + c_2(\delta_0, \theta)C_2(\varepsilon_0, \beta)(1 + \log \chi(t))} dt.$$

It follows that there is a constant  $c_2(\varepsilon_0, \delta_0, \theta, \beta)$  such that in a small enough neighbourhood  $U$  of  $(\delta_0, \lambda_c(\delta_0))$  we have

$$\frac{1}{\chi(t_1)} - \frac{1}{\chi(t_2)} \geq c_2 \int_{t_1}^{t_2} \frac{1}{1 + \log \chi(t)} dt.$$

The second part now follows from [3, Lemma 4.1].  $\square$

*Proof of Theorem 1.3.* The lower bounds are immediate from Proposition 3.3. The upper bounds follow from Proposition 3.5 on noting that  $\|(\delta, \lambda) - (\delta_0, \lambda_c(\delta_0))\| = \sqrt{1 + \theta^2} \cdot |t|$ .  $\square$

**REMARK 3.6.** A similar argument as in Proposition 3.5 would give the critical exponent value  $\gamma = 1$  also for ‘vertical’ ( $\delta$  constant) and ‘horizontal’ ( $\lambda$  constant) approach to the critical curve, subject to first proving differential inequalities of the form

$$\frac{\partial \chi_\Lambda}{\partial \lambda} \leq c_1 \left( -\frac{\partial \chi_\Lambda}{\partial \delta} \right) \quad \text{and} \quad -\frac{\partial \chi_\Lambda}{\partial \delta} \leq c_2 \frac{\partial \chi_\Lambda}{\partial \lambda},$$

with  $c_1, c_2$  uniform in  $\Lambda$ . We do not pursue this here, only noting that similar inequalities hold for classical Ising and Potts models [4, 7]. In the absence of such inequalities it is a-priori possible that the behaviour for vertical or horizontal approach differs from the case in Theorem 1.3. For example, if it were the case that  $\lambda_c(\delta) \sim (\delta - \delta_0)^2$  as  $\delta$  decreases to some  $\delta_0 > 0$ , then Proposition 3.5 would imply that  $\chi(\delta, \lambda_c(\delta_0)) \sim (\delta - \delta_0)^{-2}$ .

## REFERENCES

- [1] M. Aizenman. Geometric analysis of  $\phi^4$  fields and Ising models. *Communications in Mathematical Physics*, 86:1–48, 1982.
- [2] M. Aizenman and R. Fernández. On the critical behavior of the magnetization in high-dimensional Ising models. *Journal of Statistical Physics*, 44:393–454, 1986.
- [3] M. Aizenman and R. Graham. On the renormalized coupling constant and the susceptibility in  $\phi^4$  field theory and the Ising model in four dimensions. *Nuclear Physics B*, 225:261–288, 1983.
- [4] M. Aizenman and G. R. Grimmett. Strict monotonicity for critical points in percolation and ferromagnetic models. *Journal of Statistical Physics*, 63:817–835, 1991.
- [5] M. Aizenman, A. Klein, and C. M. Newman. Percolation methods for disordered quantum Ising models. In R. Kotecký, editor, *Phase Transitions: Mathematics, Physics, Biology*. World Scientific, Singapore, 1992.
- [6] M. Aizenman and B. Nachtergaele. Geometric aspects of quantum spin states. *Communications in Mathematical Physics*, 164:17–63, 1994.
- [7] C. E. Bezuidenhout, G. R. Grimmett, and H. Kesten. Strict inequality for critical values of Potts models and random-cluster processes. *Communications in Mathematical Physics*, 158:1–16, 1993.
- [8] M. Biskup. Reflection positivity and phase transitions in lattice spin models. In *Methods of Contemporary Mathematical Statistical Physics*, volume 1970 of *Lecture Notes in Mathematics*. Springer, Berlin, 2009.
- [9] J. E. Björnberg. *Graphical representations of Ising and Potts models*. PhD thesis, Cambridge and KTH, 2009. arXiv:1011.2683.
- [10] J. E. Björnberg and G. R. Grimmett. The phase transition of the quantum Ising model is sharp. *Journal of Statistical Physics*, 136(2):231, 2009.
- [11] N. Crawford and D. Ioffe. Random current representation for transverse field Ising model. *Communications in Mathematical Physics*, 296:447–474, 2010.
- [12] F. J. Dyson, E. H. Lieb, and B. Simon. Phase transitions in quantum spin systems with isotropic and nonisotropic interactions. *Journal of Statistical Physics*, 18(4):335–383, 1978.
- [13] J. Fröhlich, R. Israel, E. H. Lieb, and B. Simon. Phase transitions and reflection positivity I: General theory and long range lattice models. *Communications in Mathematical Physics*, 62:1–34, 1978.
- [14] J. Fröhlich, B. Simon, and T. Spencer. Infrared bounds, phase transitions and continuous symmetry breaking. *Communications in Mathematical Physics*, 50(1):79–95, 1976.
- [15] J. Ginibre. Existence of phase transitions for quantum lattice systems. *Communications in Mathematical Physics*, 14(3):205–234, 1969.
- [16] T. Hara and G. Slade. Mean-field critical behaviour for percolation in high dimensions. *Communications in Mathematical Physics*, 128:333–391, 1990.
- [17] D. Ioffe. Stochastic geometry of classical and quantum Ising models. In *Methods of Contemporary Mathematical Statistical Physics*, volume 1970 of *Lecture Notes in Mathematics*. Springer, Berlin, 2009.
- [18] E. Lieb, T. Schultz, and D. Mattis. Two soluble models of an antiferromagnetic chain. *Annals of Physics*, 16:407–466, 1961.
- [19] S. Sachdev. Quantum phase transitions. In *Handbook of Magnetism and Advanced Magnetic Materials*. John Wiley & Sons, Ltd, 2007.
- [20] A. D. Sokal. A rigorous inequality for the specific heat of an Ising or  $\phi^4$  ferromagnet. *Physics Letters*, 71A(5,6):451–453, 1979.